Thompson's Group $F(p+1)$ is not Minimally Almost Convex

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1 Thompson's Group $F(p+1)$

1.1 A Description of $F(p+1)$

Thompson's group $F(p+1)$ can be defined as the group of piecewise-linear orientation-preserving homeomorphisms of the closed unit interval with breakpoints in $\mathbb{Z}[\frac{1}{p+1}]$ and slopes which are powers of $p+1$ in each linear piece.

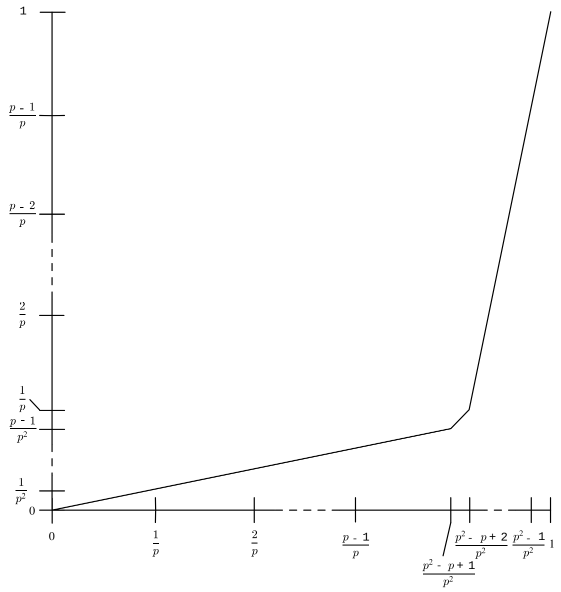

Figure 1: One element of $F(p+1)$

1.2 Representation of $F(p+1)$ by tree-pair diagrams

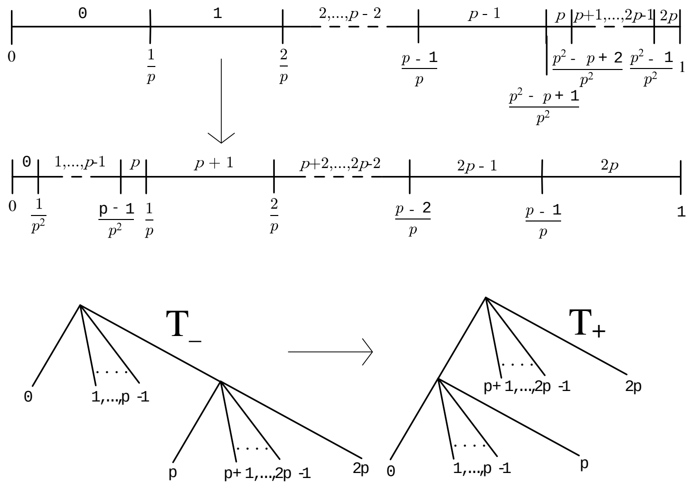

Figure 2: The homeomorphism of the closed unit interval and the tree-pair diagram representing x_0 in $F(p+1)$

1.2.1 Definitions

A caret in a $(p+1)$-ary tree has $(p+2)$ vertices. One of these vertices is called the *parent*; the parent is connected by $(n+1)$ directed edges from itself to each of the other vertices in the caret, which we call the *children* of the parent vertex. Two children of the same parent are called *siblings*. A graph formed by joining any number of $(p+1)$-ary carets is referred to as a $(p+1)$-*ary tree*. The topmost caret is referred to as the *root caret* (or just the *root*) and it's parent is called the root node. If a vertex in the tree has no children, it is referred to as a *leaf*; if it does have children, it is referred to as a *node*.

An element of $F(p+1)$ can be represented by a $(p+1)$-ary tree pair diagram. A $(p+1)$-*ary tree pair diagram* is a pair of $(p+1)$-ary trees containing the same number of nodes. The first tree in the pair is the *negative tree* and the second tree in the pair is the *positive tree*. This pair of trees is denoted (T_-, T_+). The nodes of each tree are numbered , and the n^{th} node of the negative tree is paired with the n^{th} node of the positive tree.

1.2.2 Leaf Ordering in a Tree-pair Diagram

We can number the leaves of each of the trees in a tree-pair diagram of an element of $F(p+1)$ by thinking of each leaf as a subinterval of the closed unit interval; we number the leaves of the tree from left to right with respect to their position as subintervals of the closed unit interval.

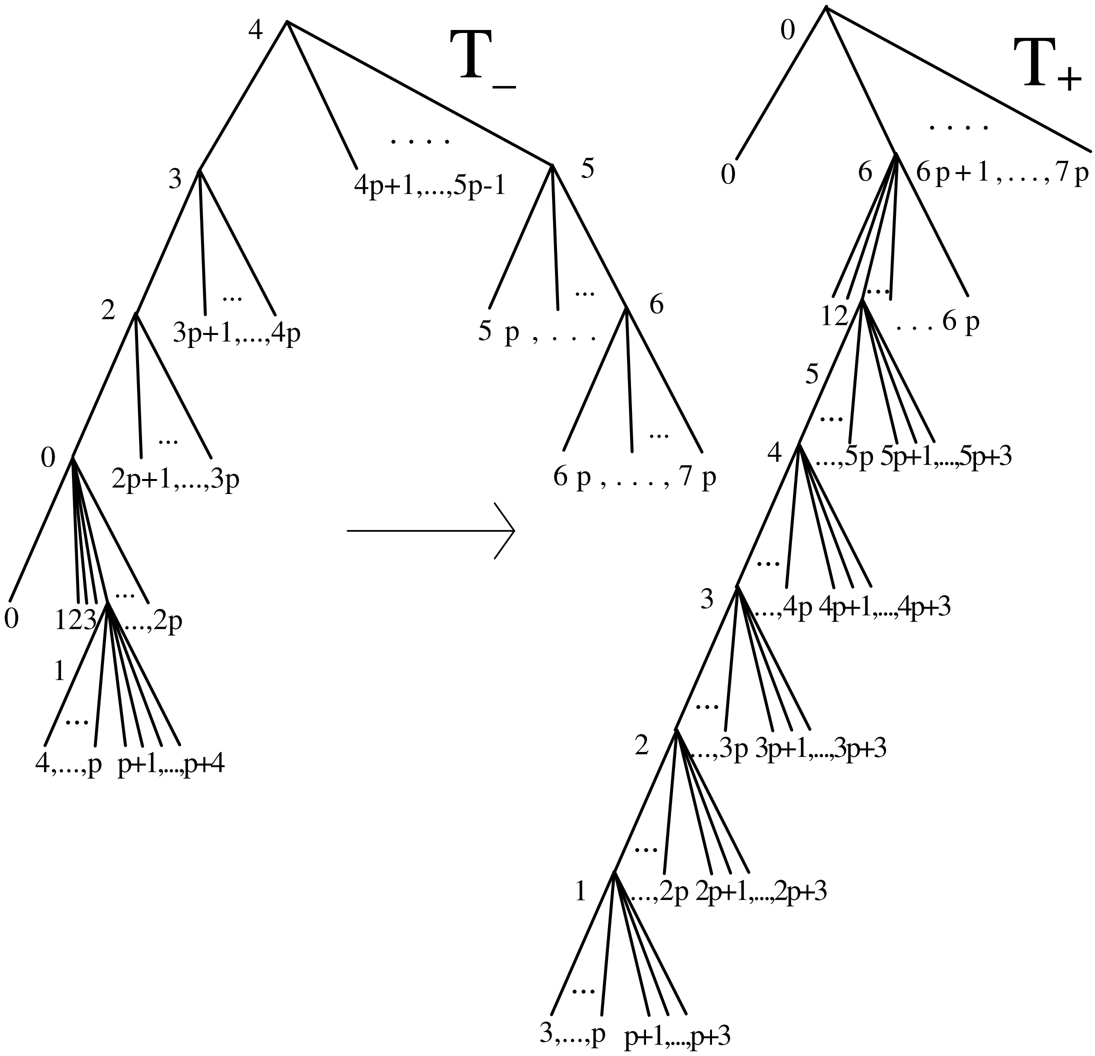

Figure 3: $x_1 x_3^5 x_4^{-1} x_0^{-3}$ in $F(p+1)$ with all carets and leaves numbered

1.3 Presentations of $F(p+1)$

$F(p+1)$ standard infinite presentation:

$$F(p+1) = \{x_0, x_1, ... | x_j x_i = x_i x_{j+p} \text{ for } i < j\}$$

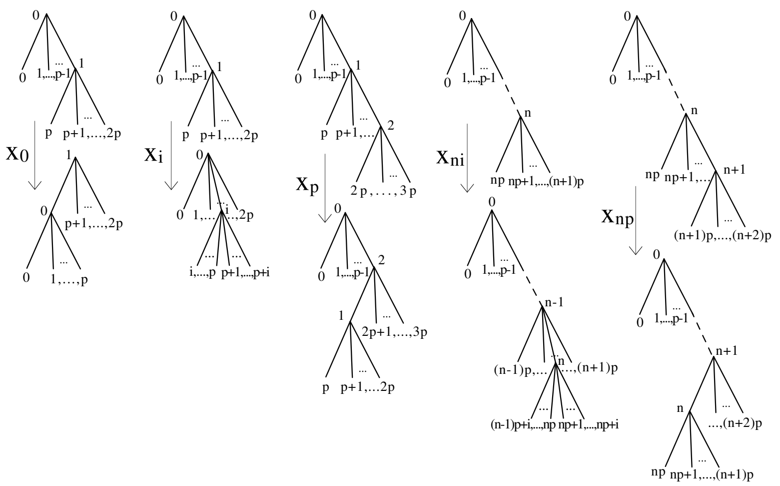

Figure 4: The generators $\{x_0, x_1, x_2, ...\}$ for the standard infinite presentation of $F(p+1)$ ($i = 1, 2, ...p-1$ and $n \in \mathbb{N}$)

$F(p+1)$ standard finite presentation:

$$F(p+1) = \{x_0, x_1, ..., x_p | [x_0 x_i^{-1}, x_k]\}$$

where $k = p+1, ..., 2p+1$ and $i = 1, ..., p$.

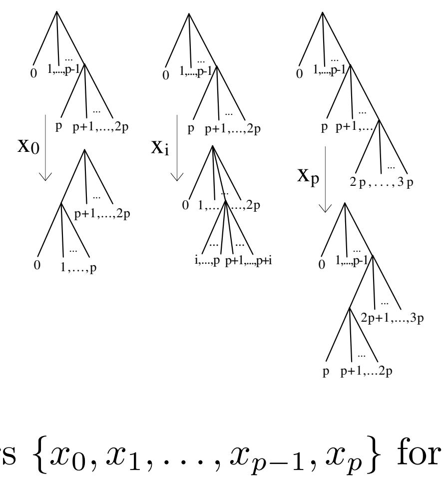

Figure 5: The generators $\{x_0, x_1, \ldots, x_{p-1}, x_p\}$ for the standard finite presentation of $F(p+1)$ ($i = 1, 2, ...p-1$)

1.4 Fordham's carat types, node/caret ordering in tree-pair diagrams, and his method for computing word length in $F(p+1)$.

Table 1: Weight of Types of Caret Pairs in the Tree-Pair Diagram of Elements in $F(p+1)$ $(j_1 \le i < j_2, i_1 < j \le i_2)$

(,)	$\mathcal{L}$	$\mathcal{R}_\emptyset$	$\mathcal{R}_{\mathrm{R}}$	$\mathcal{R}_j$	$\mathcal{M}^i_\emptyset$	$\mathcal{M}^i_j$
$\mathcal{L}$	2	1	1	1	2	2
$\mathcal{R}_\emptyset$	1	0	2	2	1	3
$\mathcal{R}_{\mathrm{R}}$	1	2	2	2	1	3
$\mathcal{R}_{j_1}$	1	2	2	2	3	3
$\mathcal{M}^{i_1}_\emptyset$	2	1	1	1	2	2
$\mathcal{M}^{i_1}_{j_1}$	2	3	3	3	4	4
$\mathcal{R}_{j_2}$	1	2	2	2	1	3
$\mathcal{M}^{i_2}_\emptyset$	2	1	1	3	2	4
$\mathcal{M}^{i_2}_{j_2}$	2	3	3	3	2	4

We will use the notation $w(\wedge_i)$ to denote the weight, given by Fordham's table, of the pair of carets in the tree-pair diagram numbered i.

We note that carets of type $\mathcal{L}_\emptyset$ are not listed on the table; since there is only one caret of this type in both the positive and negative trees, the only pairing possible is $(\mathcal{L}_\emptyset,\mathcal{L}_\emptyset)$ and $w(\mathcal{L}_\emptyset, \mathcal{L}_\emptyset) = 0$.

Theorem 1.1. *(Fordham). Given an element w in $F(p+1)$ described by the reduced tree pair diagram (T_-, T_+), the word length $|w|$ of the element with respect to the generating set $\{x_0, x_1, \ldots, x_{p-1}, x_p\}$ is the sum of the weights of each of the pairs of carets in the tree pair diagram.*

1.4.1 Multiplying tree-pair diagrams

Simplifying tree-pair diagrams:
We say that a caret is *exposed* if it has no child carets (i.e. all its children are leaves). If there exist exposed carets with exactly the same number on both the positive and negative trees then these two carets can be removed from their respective trees without changing the element that the tree-pair diagram represents. This removal of unnecessary carets is analogous to simplifying a word.

Multiplication is on the right.

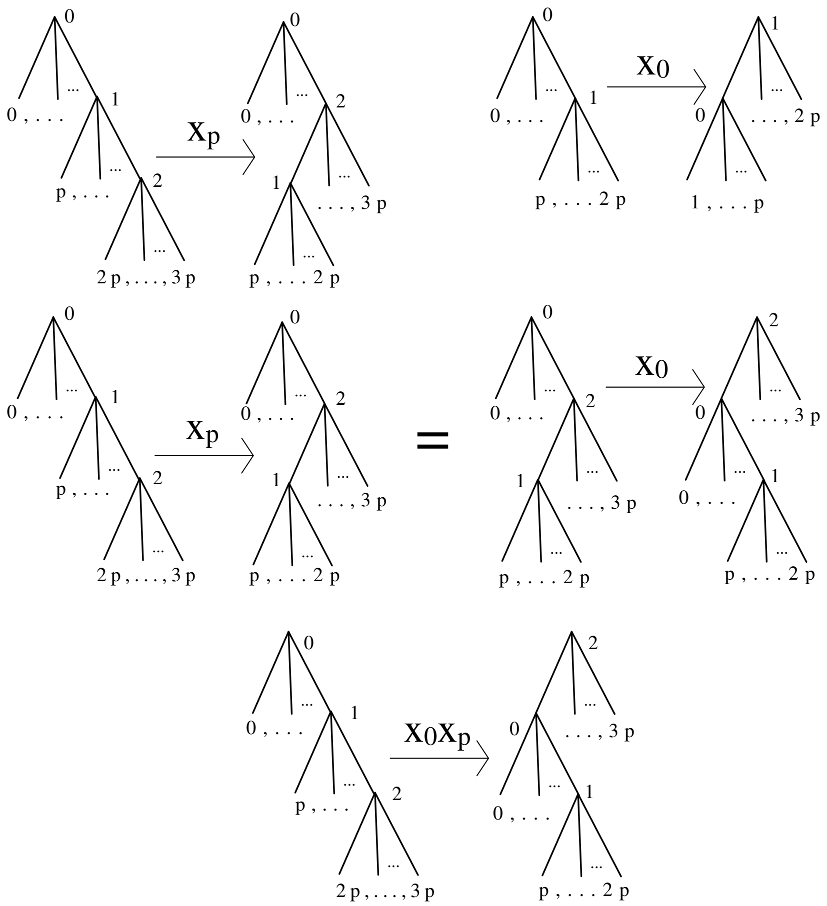

Figure 6: Multiplication of tree-pair diagrams for the product $x_0 x_p$ in $F(p+1)$ (to complete multiplication, one caret must be added to the leaf numbered p in the tree-pair diagram for x_0)

2 Almost Convexity

We let B_n denote the ball of radius n in the Cayley graph Γ of a group G with the finite generating set X. The *convexity function* $c(n)$ of the group is defined to be $c(n) = max\{d_{B_n}(g,h) | g,h \in B_n \text{ and } d_\Gamma(g,h) = 2\}$.

Definition (almost convex) 2.1. *(Cannon). A group G is almost convex with respect to the finite generating set X if its convexity function $c(n)$ with respect to the finite generating set X is bounded by a constant C, i.e. $c(n) \leq C$ for all n.*

Definition (minimally almost convex) 2.2. *A group G is minimally almost convex with respect to the finite generating set X if its convexity function $c(n)$ with respect to the finite generating set X is bounded by the constant $2n-1$ for sufficiently large n, i.e. if there exists a constant N such that for all $n > N$, $c(n) \leq 2n-1$.*

Since there is always a path from g to h in B_n through the identity which has length in B_n bounded by $2n$ ($g^{-1}h$, for example), we will always have $c(n) \leq 2n$.

Belk and Brown have already proved that $F(2)$ (usually denoted F) is not minimally almost convex; the proof that $F(p+1)$ is not minimally almost convex is a generalization of that proof.

2.1 Thompson's group $F(p+1)$ is not minimally almost convex

$L(g)$ is used to denote the *length* of an element g in the group with respect to the given generating set.

$w(\wedge_i)$ is used to denote the *weight* of the pair of carets which have index number i in the tree-pair diagram of the element.

Theorem 2.3. *For all $n \leq 1$ there exist $l, r, \in F(p+1)$ such that with respect to the standard finite generating set $\{x_0, x_1, \ldots, x_{p-1}, x_p\}$:*

1. $d(l, r) = 2$
2. $L(l) = L(r) = 2n + 2$
3. *Any path p from l to r which remains in B_{2n+2} is such that $L(p) \geq 4n + 4$.*

We choose $r = x_p^n x_0^{-(n+1)} x_p^{-1}$ and $l = r x_0^2$.

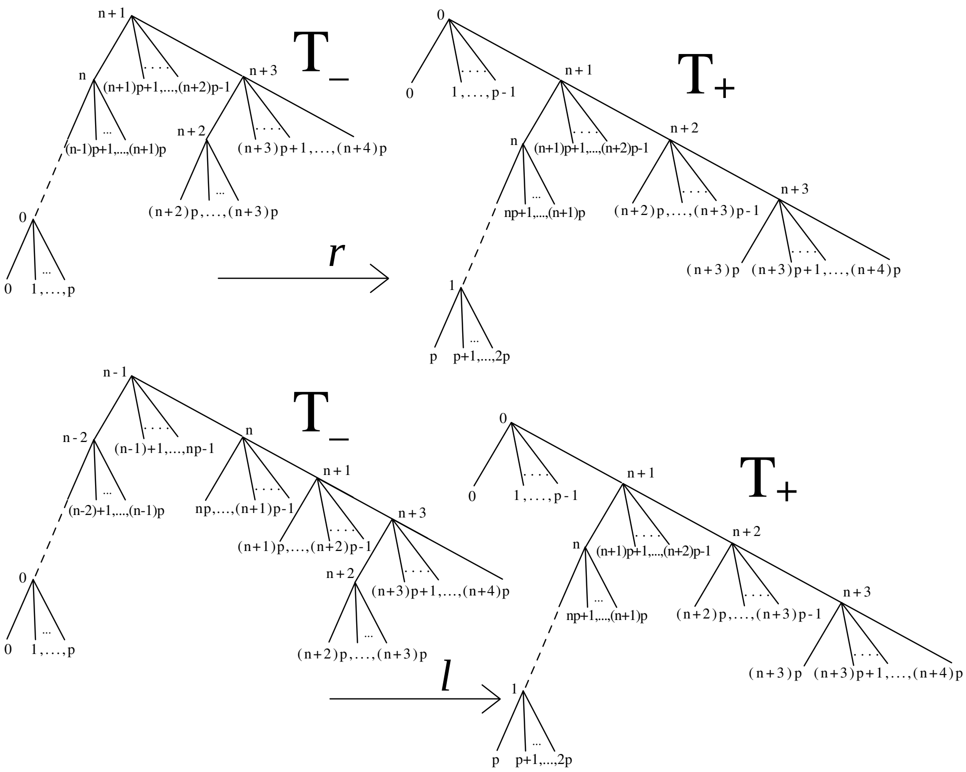

Figure 7: r (top) and l (bottom) in $F(p+1)$

2.1.1 $\exists$ 2 Vertexes on the Path from l to r with distance $\leq 2n+3$

We let p represent the path from l to r which does not leave B_{2n+2}.

Lemma 2.4. *(Belk and Bux) With respect to the standard finite generating set for $F(p+1)$, if there are two vertices h_r and h_l on the path p from l to r such that $d(h_l, h_r) \geq 2n+3$, then $L(p) \geq 4n+4$.*

Definition (right foot) 2.5. *Let $\wedge_s$ be the first right caret in the negative tree.*

1. *If $\wedge_s$ has a nonempty left subtree, then we consider this subtree: within this new subtree, the right foot is the rightmost child of the last right caret in the subtree, which will be a leaf (if it were not a leaf, there would be another right caret in the tree).*
2. *If $\wedge_s$ has an empty left subtree, then the right foot is the leftmost leaf of $\wedge_s$ in the negative tree.*
3. *If the negative tree has no carets of type $\mathcal{R}$, then the right foot is the rightmost leaf of the root caret in the negative tree.*

Definition (critical leaf) 2.6. *The critical leaf is the same thing as the right foot; the only difference is that the right foot is a leaf on the negative tree and the critical leaf is a leaf on the positive tree.*

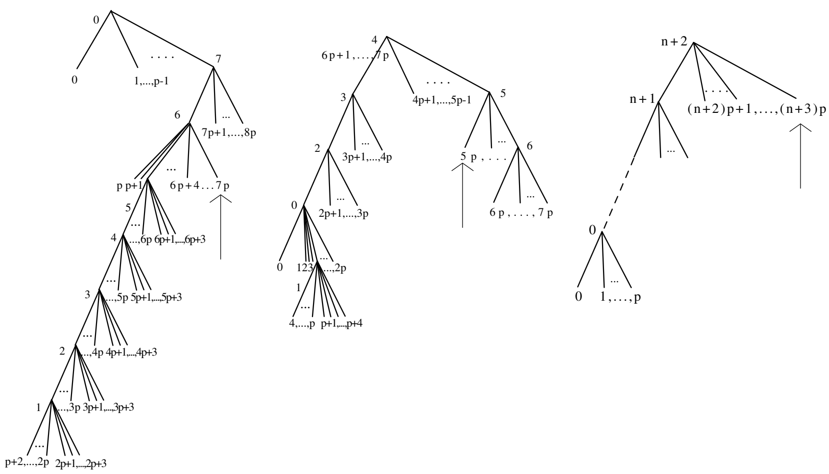

Figure 8: Right foot/critical leaf indicated by arrow in several $(p+1)$-ary trees

For the element l, the right foot is to the left of the critical leaf (index of right foot is np and index of critical leaf is $(n+1)p$), and for the element r, the right foot is to the right of the critical leaf (index of right foot is $(n+3)p$ and index of critical leaf is $(n+1)p$).

On the path p from l to r:
The right foot *crosses* the critical leaf once we have the index of the the right foot greater than or equal to the index of the critical leaf. The right foot *hits* the critical leaf when they both have the same index number.

2.1.2 Finding the vertex h_r

Lemma 2.7. *Any path p from l to r which remains in B_{2n+2} must pass through the vertex rx_px_0.*

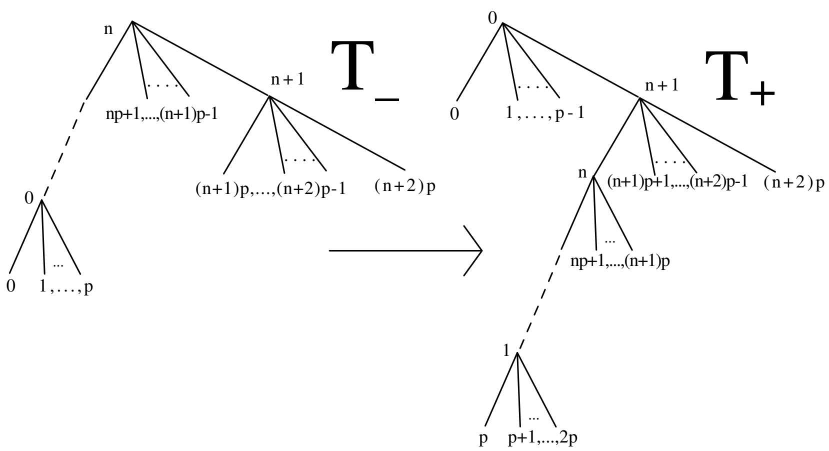

Figure 9: h_r in $F(p+1)$

If we let $h_r = rx_px_0$, h_r is the last vertex along p on which the right foot hits the critical leaf.

2.1.3 Finding the vertex h_l

We let h_l be the first vertex along p on which the right foot hits the critical leaf.

Lemma 2.8. *For any vertex on the path p from l to r on which the right foot has not yet crossed the critical leaf, the last 3 carets in the positive and negative trees of that vertex will remain the same as they were in l (see figure 10). These are the carets which have an index number higher than the caret containing the critical leaf.*

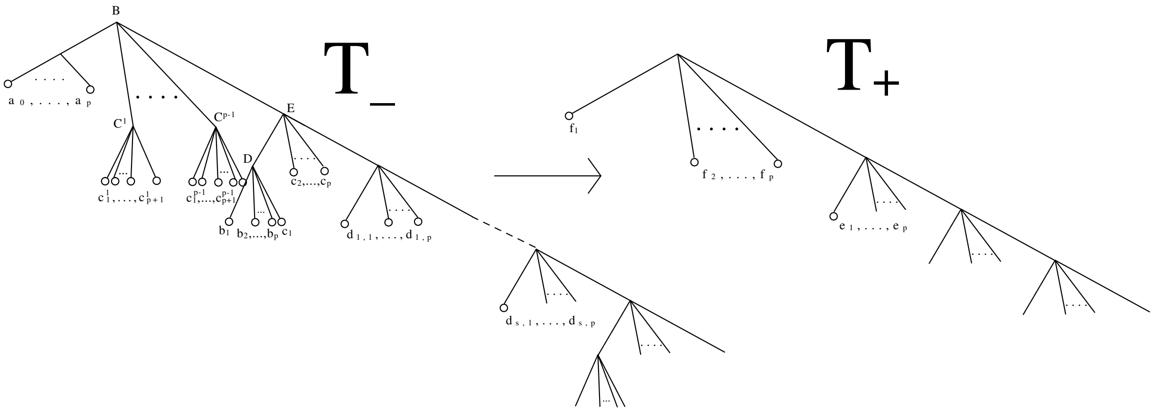

Figure 10: A vertex on the path p from l to r at which the right foot has not yet crossed the critical leaf (after carets have been added as needed so that it can be multiplied by each of the generators)

Lemma 2.9. *The first vertex on the path p from l to r at which the right foot crosses the critical leaf is also the first vertex at which the right foot hits the critical leaf (i.e. the right foot cannot cross the critical leaf for the first time on the path without hitting the critical leaf).*

2.1.4 Left-Sided Elements

Definition (left-sided) 2.10. *An element f of $F(p+1)$ is left-sided if:*

1. *The index number of the right foot is the same as the index number of the critical leaf (i.e. $t = s$), and*
2. *All right carets in each tree are of type $\mathcal{R}_\emptyset$.*

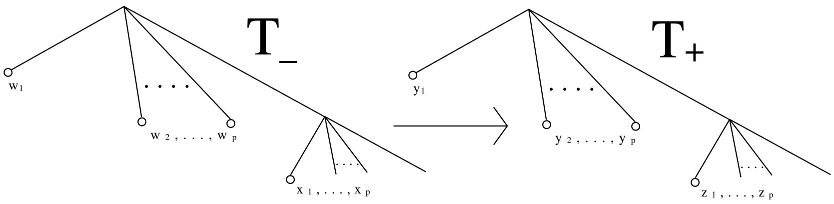

Figure 11: A left-sided element in $F(p+1)$

Remark 2.11. *h_r is left-sided.*

Definition 2.12. *We define $y = x_0^{-1} x_p x_0$.*

Remark 2.13. *Left-sided elements commute with y.*

Lemma 2.14. *For any left-sided element f in $F(p+1)$, $L(fy) = L(f)+3$.*

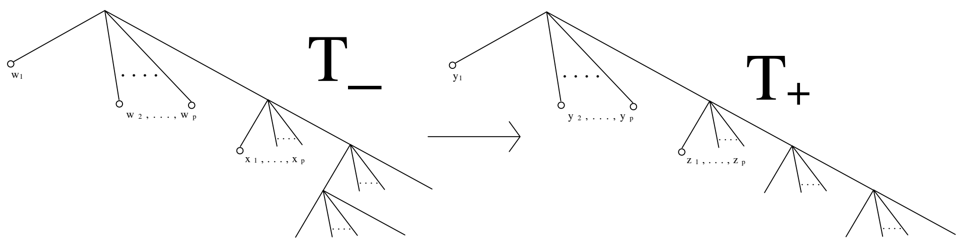

Figure 12: $yf = fy$ for left-sided element f in $F(p+1)$

Lemma 2.15. *There exists a left-sided element* h'_l *on the path* p *from* l *to* r *such that* $h_l = yh'_l$*.*

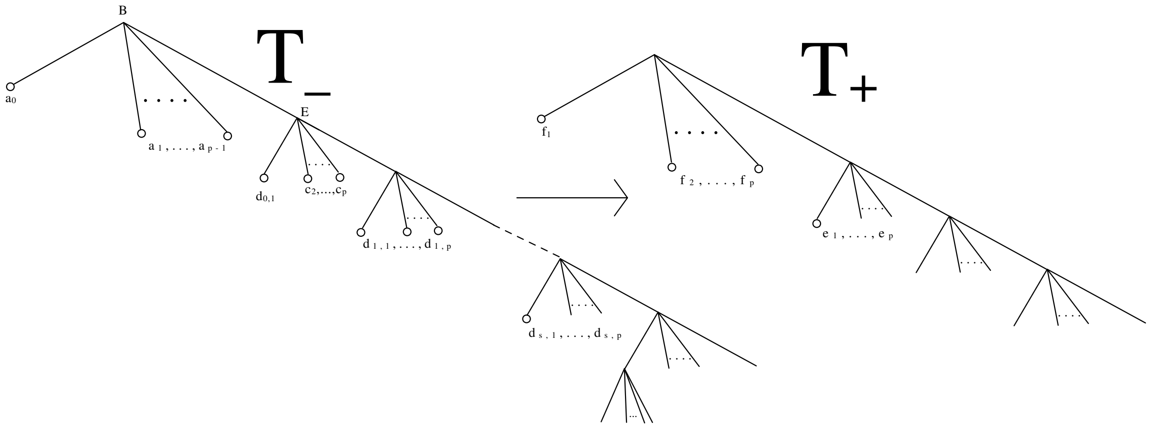

Figure 13: Simplified version of a vertex on the path p from l to r at which the right foot has not yet crossed the critical leaf

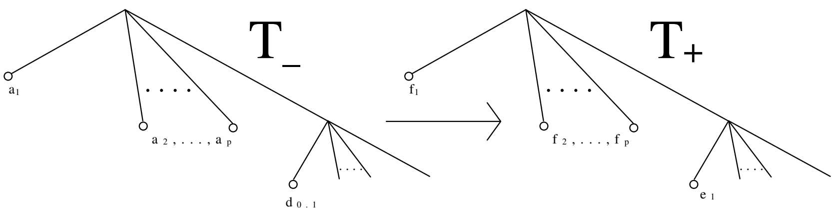

Figure 14: h'_l in $F(p+1)$

Lemma 2.16. *The tree-pair diagram of $h_r^{-1}h_l'$ is already reduced.*

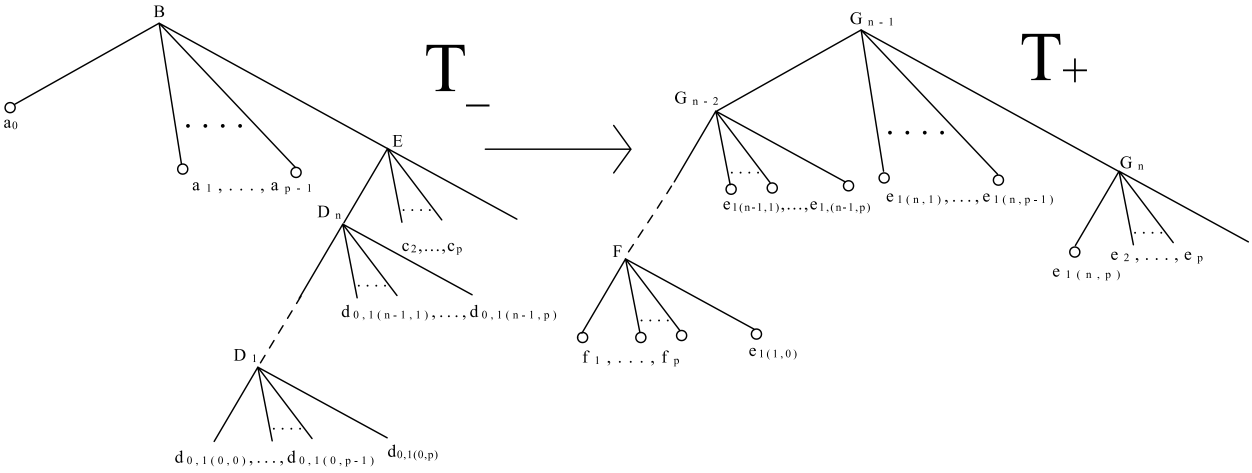

Figure 15: $h_r^{-1}h_l'$ in $F(p+1)$

Definition (width of a group element) 2.17. *Let m denote the number of carets in either the positive or negative tree of the tree pair diagram of f in $F(p+1)$. The width of f, denoted by $w(f)$, is equal to $m-2$.*

Lemma 2.18. *For any left-sided element f of $F(p+1)$, $L(f) \geq 2w(f)$.*

Remark 2.19. *$h_r^{-1}h_l'$ is left-sided.*

Corollary 2.20. $L(h_r^{-1}h_l') \geq 2n$

Theorem 2.21. *$F(p+1)$ is not minimally almost convex.*