A Normal Form for Thompson's Group $F(n, m)$

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Outline of Results

1. Seesaw words exist in $F(n)$ with respect to the standard finite generating set and therefore $F(n)$ is not combable by geodesics and there does not exist a regular language of geodesics for $F(n)$.

2. Dead ends exist in $F(n)$ with respect to the standard finite generating set, and all dead ends have depth 2.

3. $F(n)$ is not minimally almost convex with respect to the standard finite generating set.

4. Tree-pair diagrams can be used to represent elements of $F(n, m)$ just as in the case of $F(n)$, but there are several key differences; in $F(n, m)$, minimal tree-pair diagram representatives may not be unique, carets may need to be added in order to obtain a minimal tree-pair diagram representative, and there will be representatives of the identity which do not have identical domain and range trees.

5. A unique normal form exists for $F(2, 3)$ with respect to the standard infinite generating set, and this normal form can be translated directly from a minimal tree-pair diagram representative of an element and vice versa.

6. The metric on $F(2, 3)$ with respect to the standard finite generating set, unlike $F(2)$, is not quasi-isometric to the number of carets or leaves in a minimal tree-pair diagram representative of a given element. In fact, while a sharp upper bound exists on the metric which is linear with respect to the number of leaves or carets in the minimal tree-pair diagram representative, a sharp lower bound exists which is logarithmic with respect to the number of leaves or carets in the minimal tree-pair diagram representative of a given element.

Preliminary Definitions

We let $n = p + 1$ and $m = q + 1$ for $p, q \in \mathbb{N}$.

An $(p+1)$–*ary caret* has $(p+2)$ vertices, one with degree $p+1$ (the *parent*) and the rest with degree 1 (the *children*).

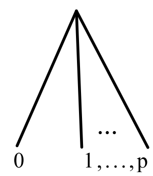

Figure 1: A $(p+1)$–ary caret

A graph formed by joining any number of $(p+1)$-ary carets is referred to as a $(p+1)$-*ary tree*.

A graph formed by joining any number of $(p+1)$–ary in combination with any number of $(q+1)$–ary carets is referred to as a $(p+1, q+1)$–*ary tree*.

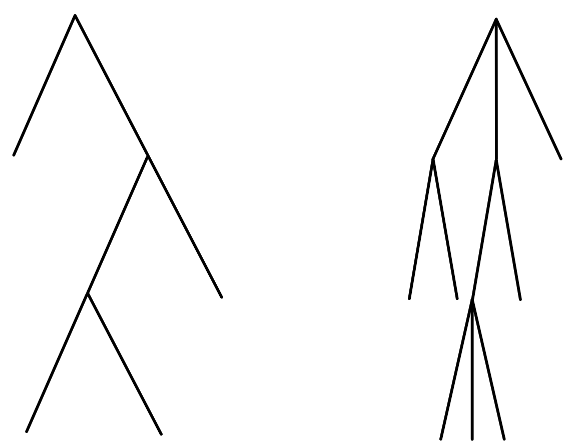

Figure 2: A 2–ary tree (left) and a $(2,3)$–ary tree (right)

The topmost caret in a tree is referred to as the *root*.

If a vertex in the tree has degree 1, it is referred to as a *leaf*.

An $(p+1)$-*ary tree-pair diagram* or $(p+1, q+1)$–ary tree-pair diagram is a pair of $(p+1)$-ary or $(p+1, q+1)$–ary trees respectively containing the same number of leaves. The first tree in the pair is the *negative tree* and the second tree in the pair is the *positive tree*. This pair of trees is denoted (T_-, T_+).

A Description of Thompson's Group $F(p+1)$

Thompson's group $F(p+1)$ (where $n \in \mathbb{N}$) can be defined as the group of piecewise-linear orientation-preserving homeomorphisms of the closed unit interval with fixed endpoints and finitely-many breakpoints in $\mathbb{Z}[\frac{1}{p+1}]$ and slopes in $\langle p+1 \rangle$.

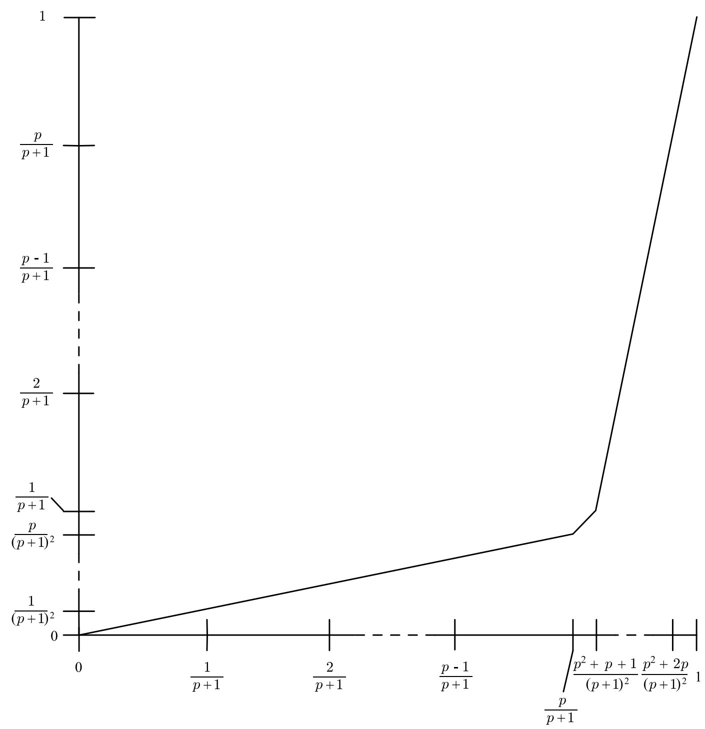

Figure 3: One element of $F(p+1)$

Representation of $F(p+1)$ by tree-pair diagrams

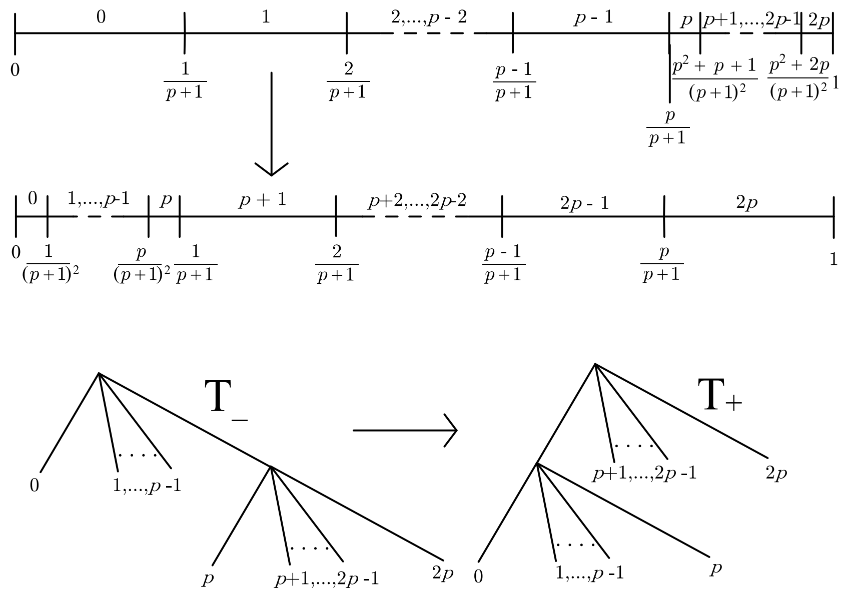

Figure 4: The homeomorphism of the closed unit interval and the tree-pair diagram representing x_0 in $F(p+1)$

Thompson's group $F(p+1, q+1)$

Definition (Thompson's group $F(p+1, q+1)$) 0.1. *The group $F(p+1, q+1)$ (where $n, m \in \mathbb{N}$) is the group of piecewise-linear orientation-preserving homeomorphisms of the unit interval with fixed endpoints, finitely-many breakpoints in $\mathbb{Z}[\frac{1}{(p+1)(q+1)}]$ and slopes in $\langle p+1, q+1 \rangle$.*

These groups have been studied from a cohomological perspective by Melanie Stein in Groups of piecewise linear homeomorphisms. *Trans. Amer. Math. Soc.*, 1992.

We consider only the cases in which $p+1$ and $q+1$ are relatively prime.

Representation of $F(p+1, q+1)$ by tree-pair diagrams

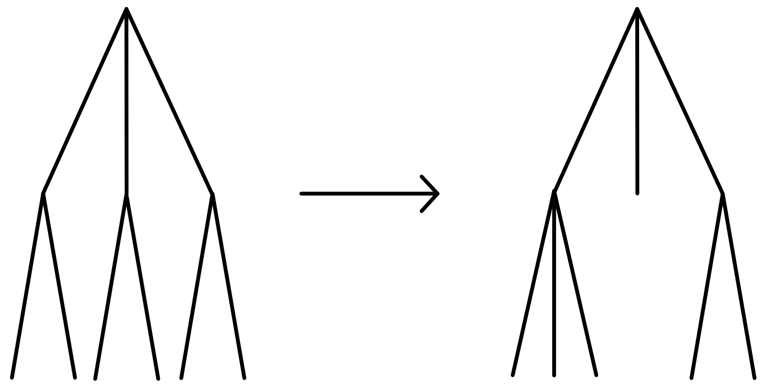

Figure 5: A $(2, 3)$–ary tree-pair diagram, which represents an element of $F(2, 3)$

Equivalent trees and Minimal tree-pair diagrams

We say that a caret is *exposed* if all its children have degree 1. Any pair of exposed carets whose leaves have identical index numbers in T_- and T_+ can be removed from their respective trees without changing the equivalence class of the tree-pair diagram. In this way we can reduce the number of leaves in a tree-pair diagram.

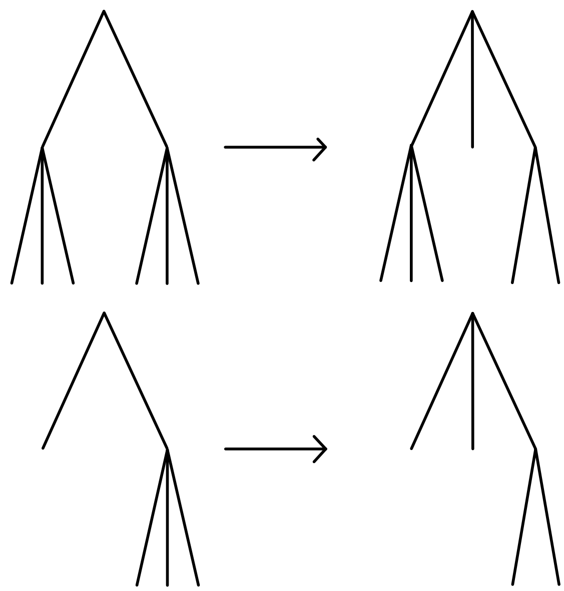

Figure 6: This $(2,3)$–ary tree-pair diagram is reduced by removing the first 3-ary caret in each tree

Definition (equivalent trees and tree-pair diagrams) 0.1. *Two $(p+1)$–ary or $(p+1, q+1)$–ary trees are equivalent if they represent the same subdivision of the unit interval. Two $(p+1)$–ary or $(p+1, q+1)$–ary tree-pair diagrams are equivalent if they represent the same element of $F(p+1)$ or $F(p+1, q+1)$ respectively.*

Definition (minimal/reduced tree-pair diagram) 0.2. *In $F(p+1)$ or $F(p+1, q+1)$, a tree-pair diagram is minimal if it has the smallest number of leaves of any tree-pair diagram in the equivalence class of tree-pair diagrams representing a given element of $F(p+1)$ or $F(p+1, q+1)$ respectively.*

Some Differences Between Tree-Pair Diagrams for $F(p+1)$ and $F(p+1,q+1)$

There may be non-obvious representatives of the identity

Tree-pair diagrams representing the identity in $F(p+1)$ will always consist of two identical trees. This is not the case in $F(p+1,q+1)$.

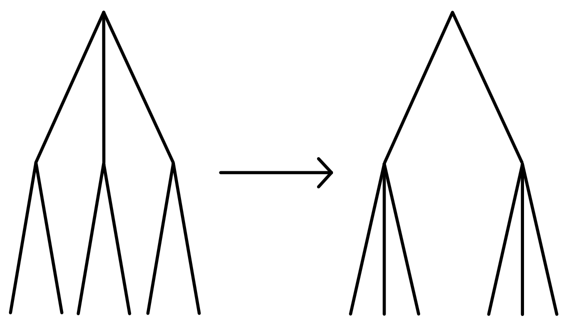

Figure 7: One tree-pair diagram for the identity element in $F(2,3)$

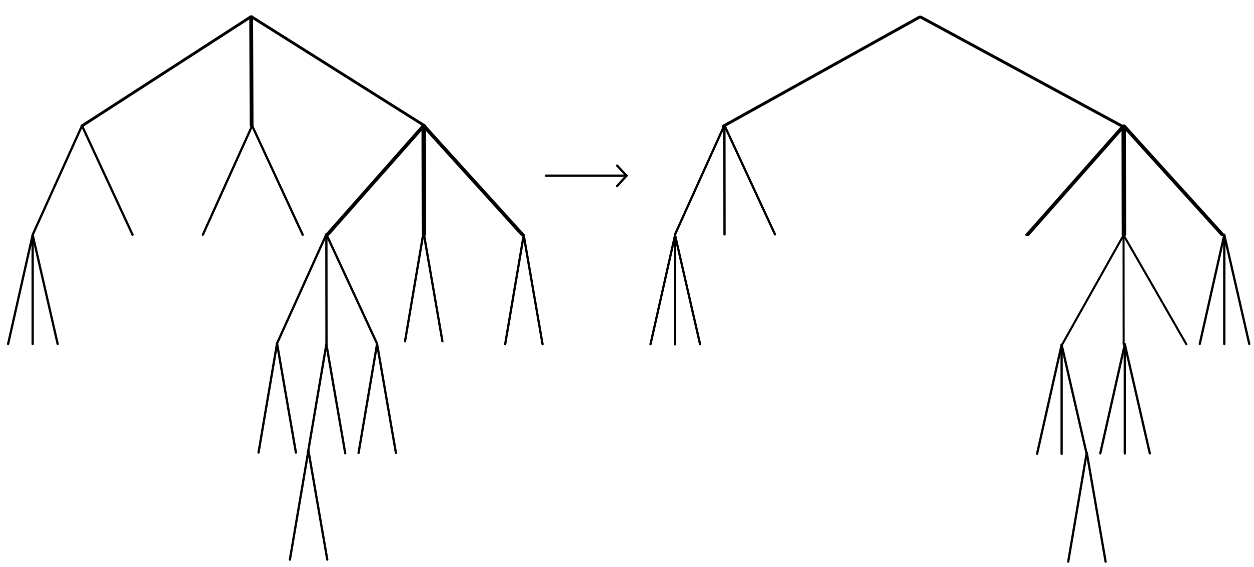

Figure 8: Another tree-pair diagram for the identity element in $F(2,3)$

Minimal tree-pair diagram representatives may not be unique

Minimal tree-pair diagram representatives of $F(p+1)$ are unique. This is not the case in $F(p+1, q+1)$.

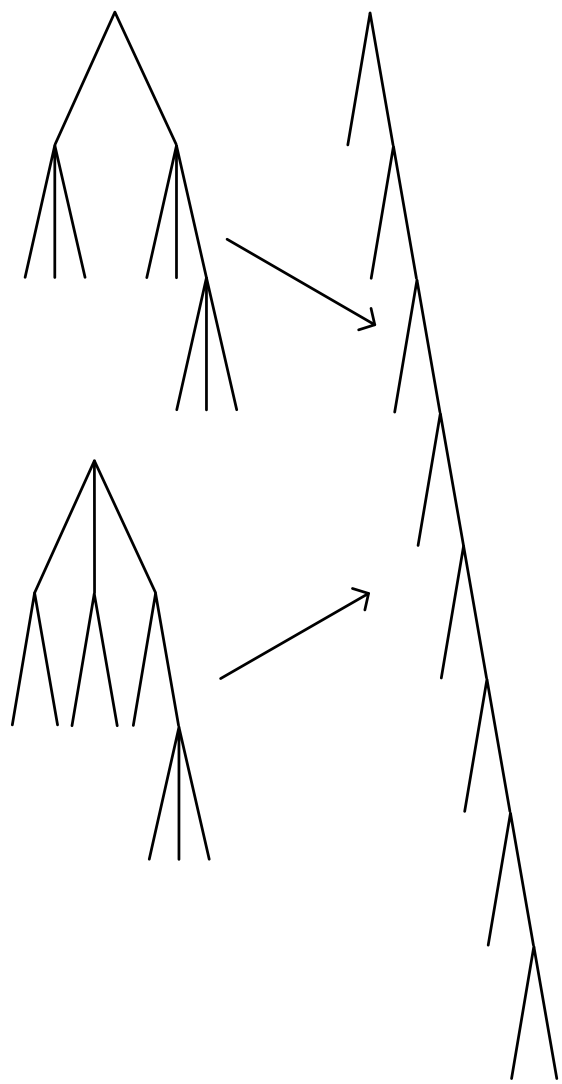

Figure 9: Two equivalent but distinct minimal tree-pair diagrams representing an element of $F(2,3)$

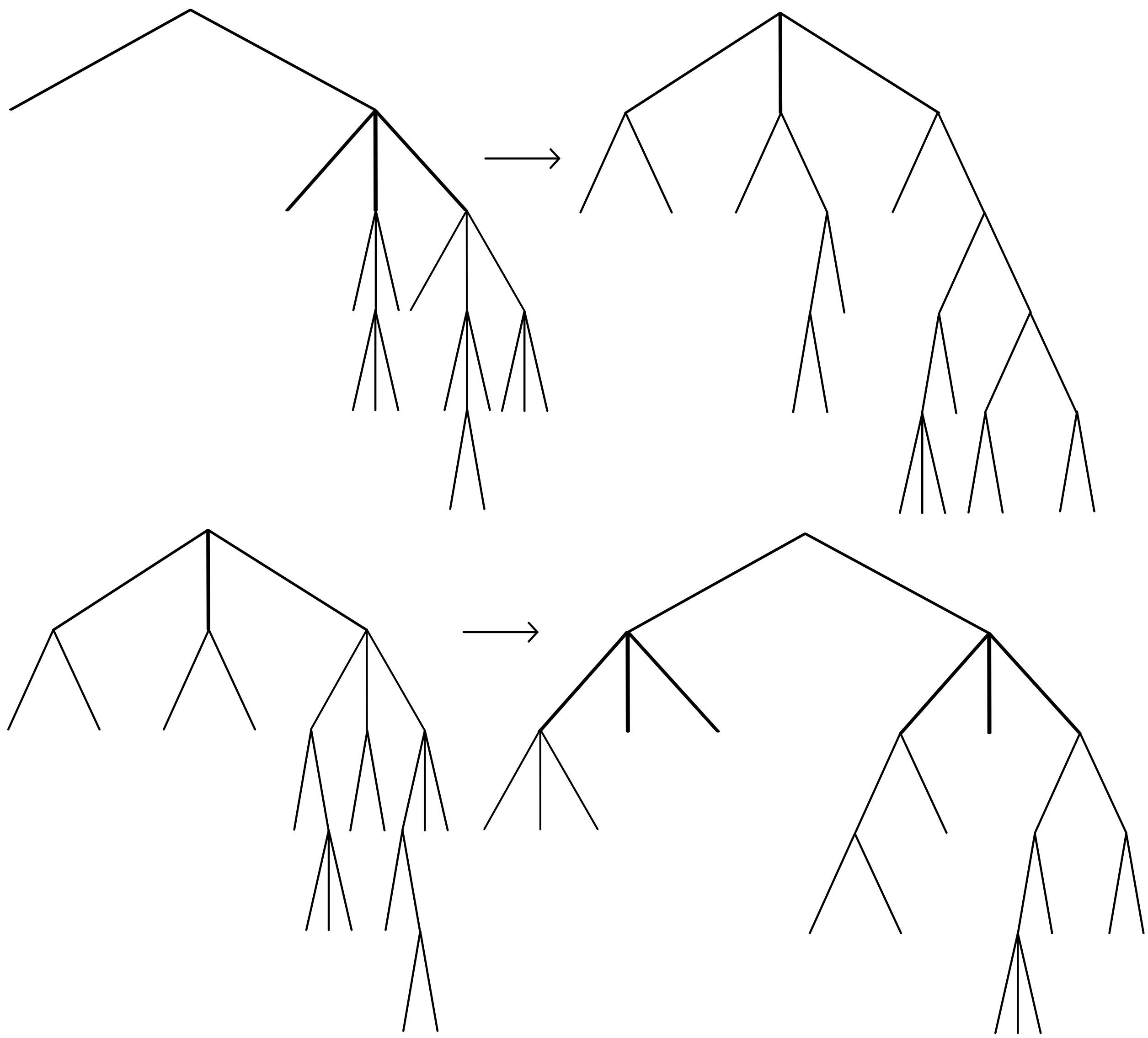

Figure 10: Two equivalent but distinct minimal tree-pair diagrams representing another element of $F(2,3)$

To get to the minimal tree-pair diagram, we may have to add carets

Minimal tree-pair diagram representatives of $F(p+1)$ can always be obtained solely by caret removal. Whereas in $F(p+1, q+1)$, we may even need to add carets in order to obtain a minimal tree-pair diagram from a given tree-pair diagram.

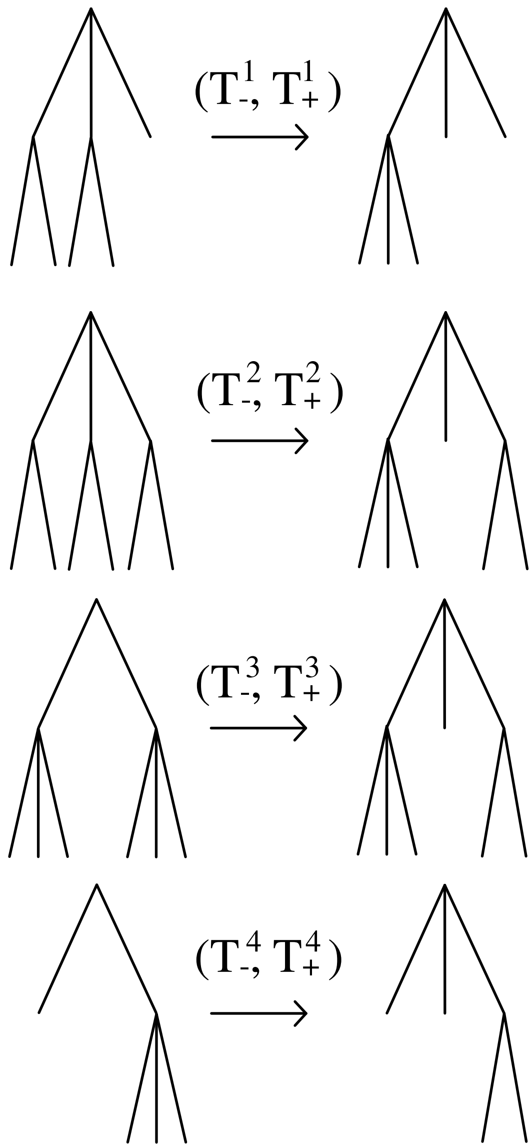

Figure 11: (T^1_-, T^1_+) is a (2,3)–ary tree-pair diagram which must have carets added to it in order to be transformed into its equivalent minimal tree-pair diagram (T^4_-, T^4_+)

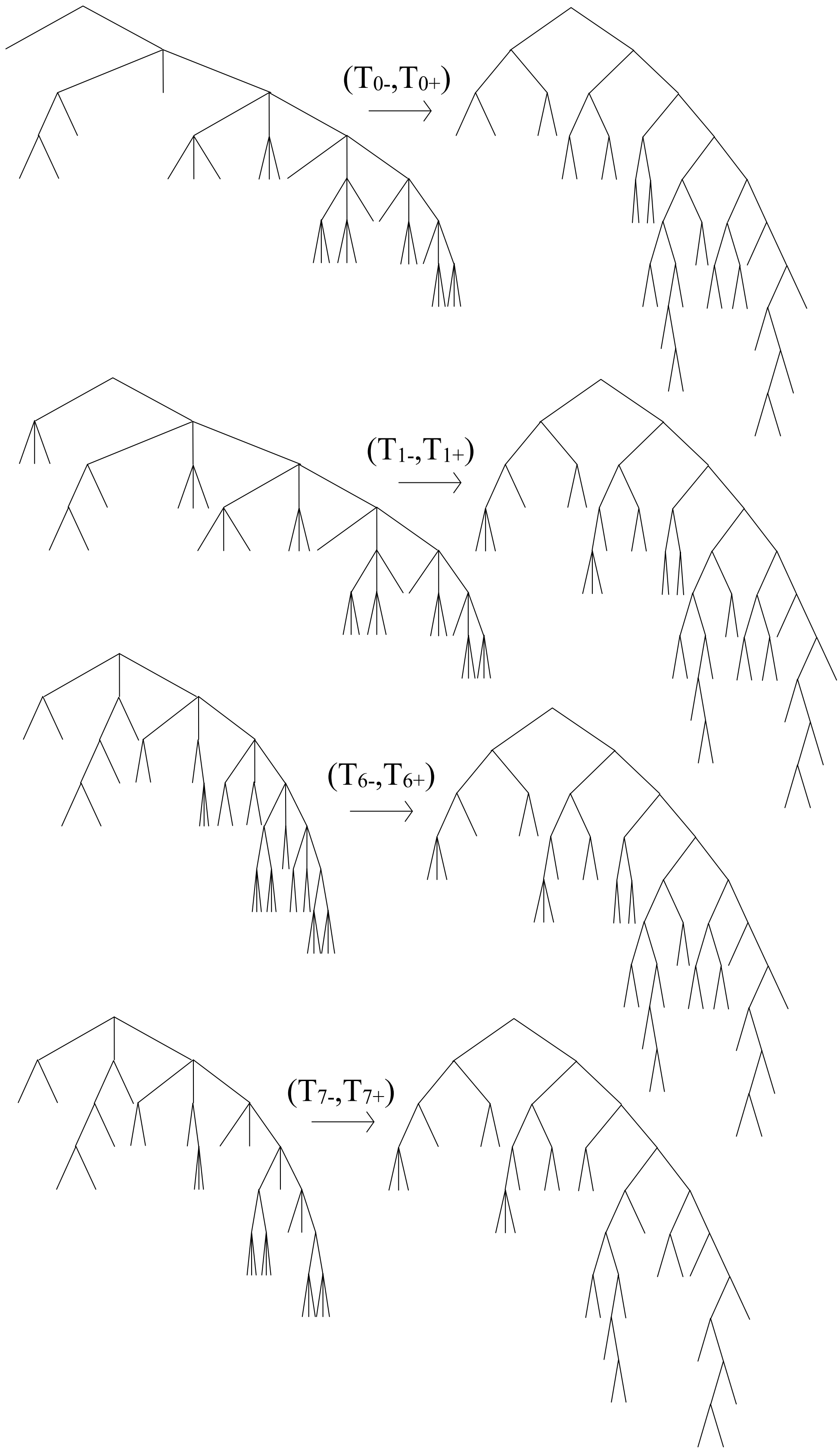

Figure 12: (T_{0-}, T_{0+}) is a (2,3)–ary tree-pair diagram which must have carets added to it in order to be transformed into its equivalent minimal tree-pair diagram (T_{7-}, T_{7+})

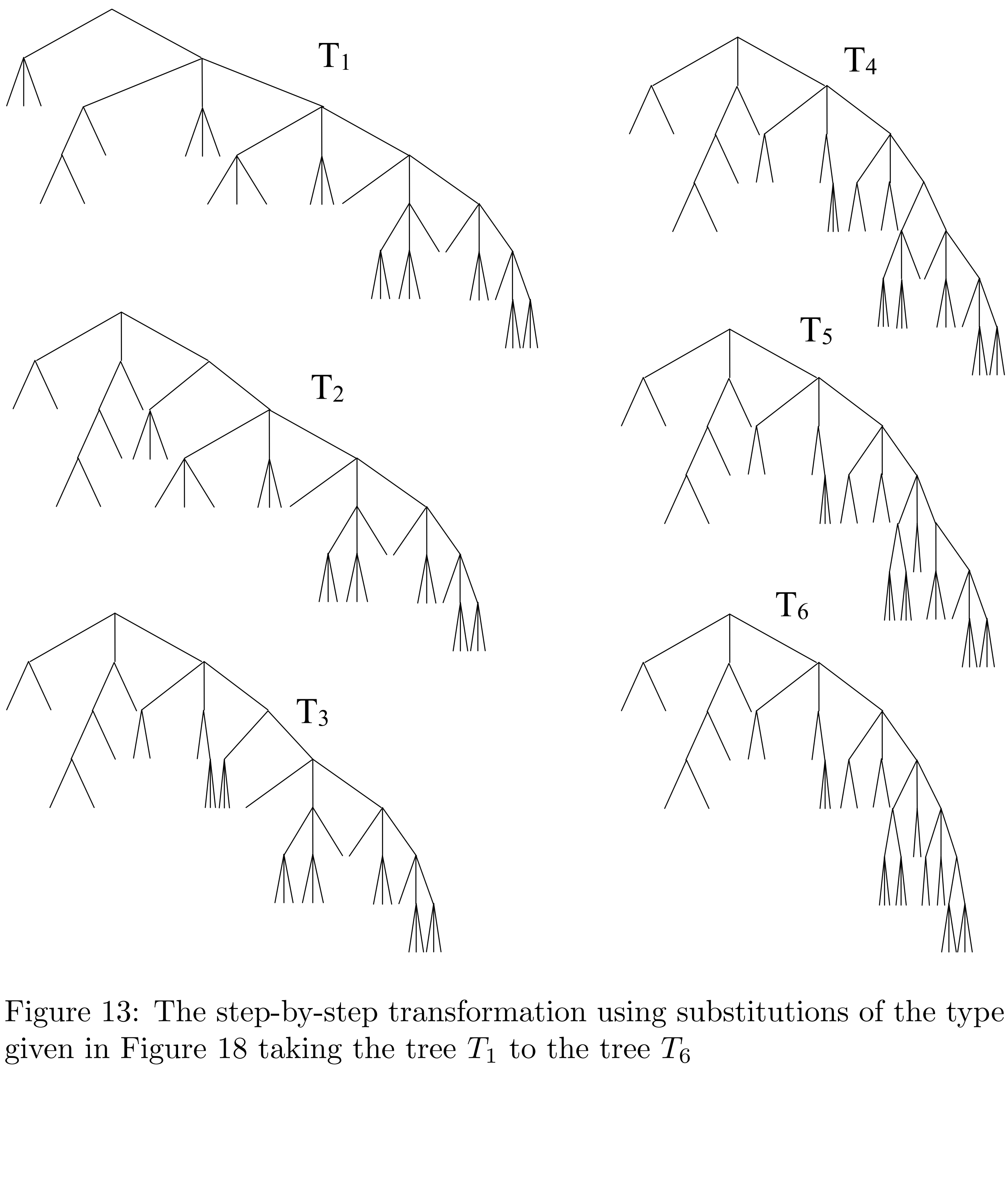

Figure 13: The step-by-step transformation using substitutions of the type given in Figure 18 taking the tree T_1 to the tree T_6

Definition (leaf-path valence) 0.3. *The n–valence of a leaf (where $n \in \mathbb{N} - \{1\}$) is the number of n–ary carets with an edge on the path in the tree from the leaf to the root node and is denoted by $v_n(\gamma_i)$. In a $(p+1, q+1)$–ary tree, the valence is the vector $\langle v_{p+1}(l_i), v_{q+1}(l_i) \rangle$.*

Theorem 0.4. *The $(p+1, q+1)$–ary tree T is equivalent to the $(p+1, q+1)$–ary tree S if and only if the number of leaves in T and S is the same and $\mathbf{v}(l_i) = \mathbf{v}(k_i)$ for all leaves l_i in T and k_i in S.*

Definition (leaf-path chart) 0.5. *A leaf path chart is a chart which is indexed by the index numbers of the leaves in a tree-pair diagram and which lists the valence of all leaves in both the positive and negative trees.*

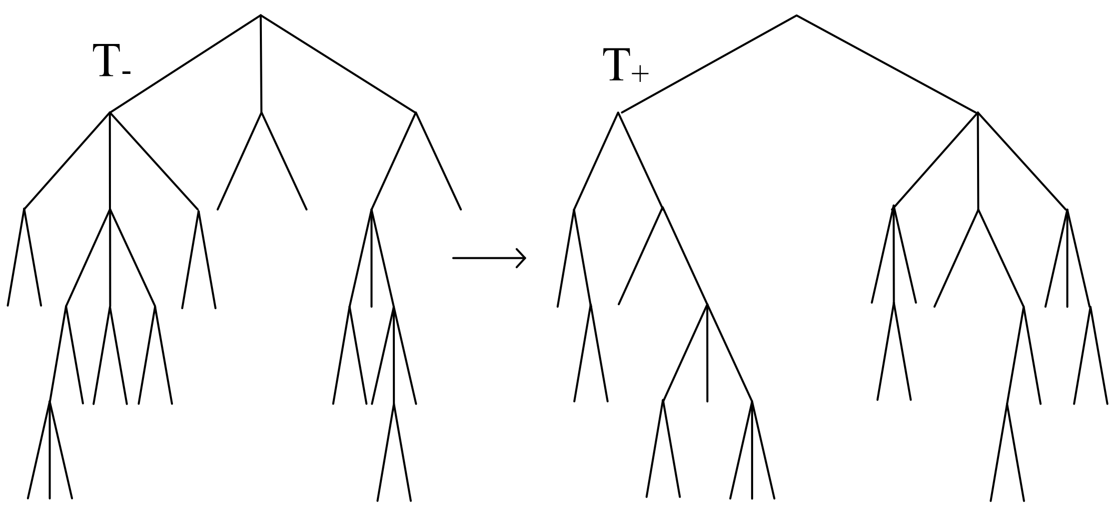

Figure 14: A (2,3)–ary tree-pair diagram

Table 1: Leaf-path chart for tree-pair diagram given in Figure 14

leaf index number	negative valence	positive valence
0	$\langle 1,2\rangle$	$\langle 3,0\rangle$
1	$\langle 1,2\rangle$	$\langle 4,0\rangle$
2	$\langle 1,4\rangle$	$\langle 4,0\rangle$
3	$\langle 1,4\rangle$	$\langle 3,0\rangle$
4	$\langle 1,4\rangle$	$\langle 4,1\rangle$
5	$\langle 1,3\rangle$	$\langle 4,1\rangle$
6	$\langle 1,3\rangle$	$\langle 3,1\rangle$
7	$\langle 1,3\rangle$	$\langle 3,2\rangle$
8	$\langle 1,3\rangle$	$\langle 3,2\rangle$
9	$\langle 1,3\rangle$	$\langle 3,2\rangle$
10	$\langle 1,2\rangle$	$\langle 1,2\rangle$
11	$\langle 1,2\rangle$	$\langle 2,2\rangle$
12	$\langle 1,1\rangle$	$\langle 2,2\rangle$
13	$\langle 1,1\rangle$	$\langle 1,2\rangle$
14	$\langle 2,2\rangle$	$\langle 2,1\rangle$
15	$\langle 2,2\rangle$	$\langle 2,1\rangle$
16	$\langle 1,2\rangle$	$\langle 3,2\rangle$
17	$\langle 1,3\rangle$	$\langle 3,2\rangle$
18	$\langle 2,3\rangle$	$\langle 1,2\rangle$
19	$\langle 2,3\rangle$	$\langle 1,2\rangle$
20	$\langle 1,3\rangle$	$\langle 2,2\rangle$
21	$\langle 1,1\rangle$	$\langle 2,2\rangle$

Multiplication of tree-pair diagrams

Multiplying tree-pair diagrams in $F(p+1)$

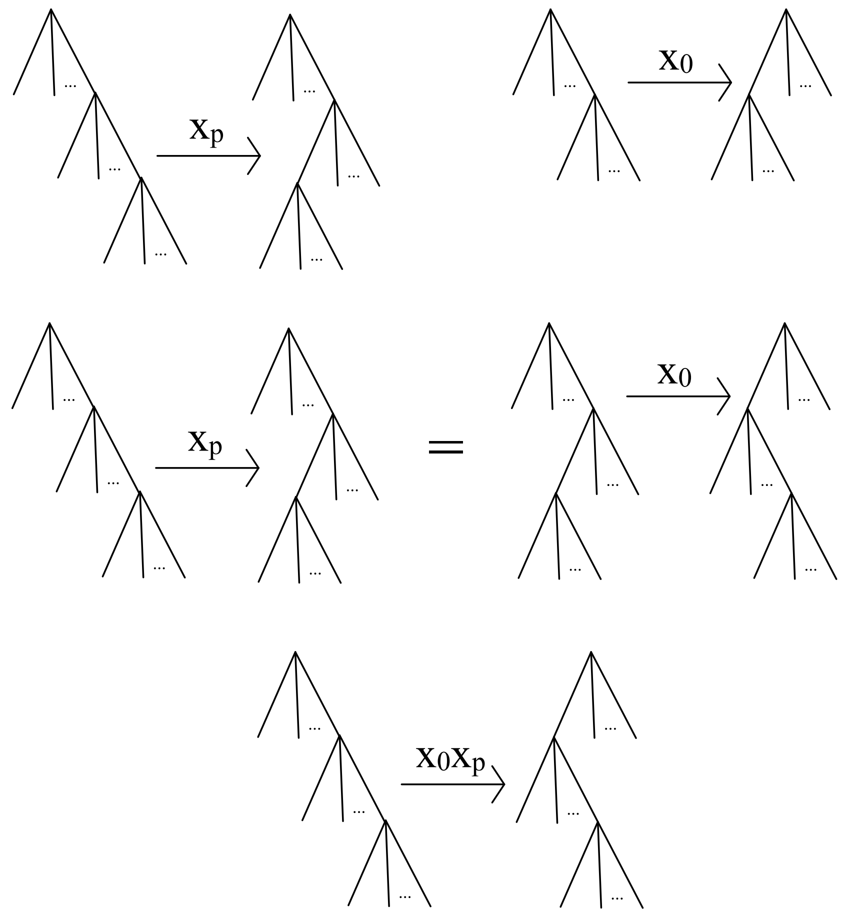

Figure 15: Multiplication of tree-pair diagrams for the product x_0x_p in $F(p+1)$ (to complete multiplication, one caret must be added to the leaf numbered n in the tree-pair diagram for x_0)

Multiplying tree-pair diagrams on $F(p+1, q+1)$

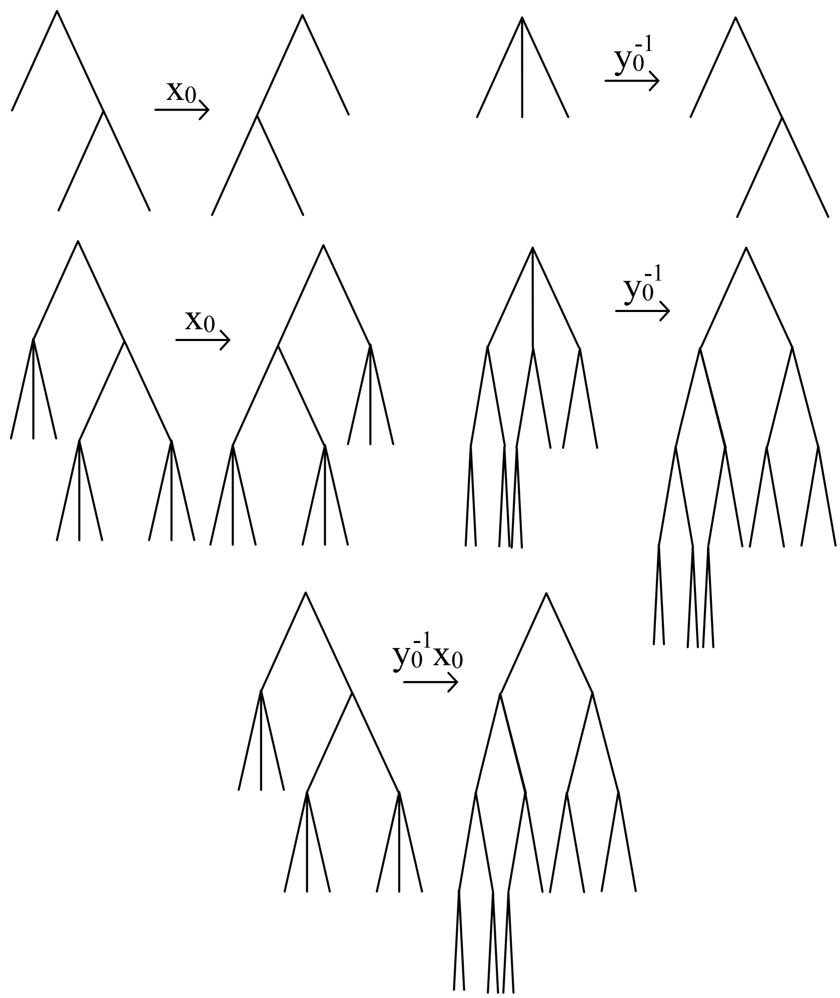

Figure 16: Composition of two elements of $F(2, 3)$

Theorem 0.6. *Any tree can be transformed into any other equivalent tree by a sequence of equivalent subtree replacements of the type indicated by the two equivalent trees given in Figure 17.*

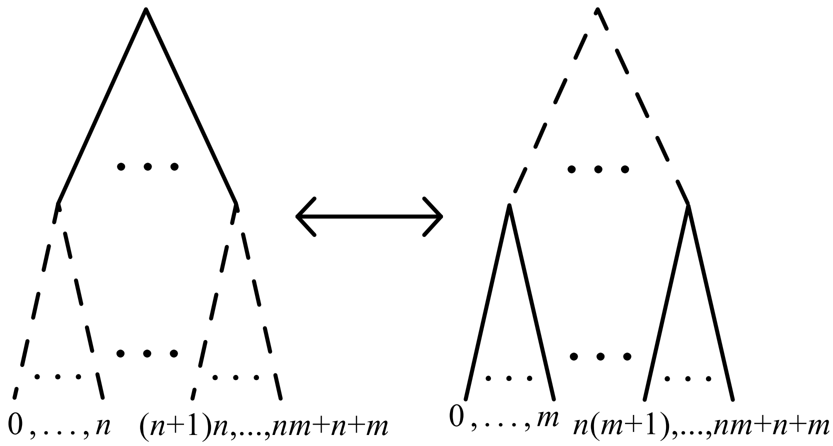

Figure 17: The two equivalent subtrees of Theorem 0.6. Here the carets composed of dotted lines are $(p+1)$–ary and the carets composed of solid lines are $(q+1)$–ary.

For example, in the case when $n = 1$ and $m = 2$, the equivalent subtree replacement diagram would be of the type given in Figure 18.

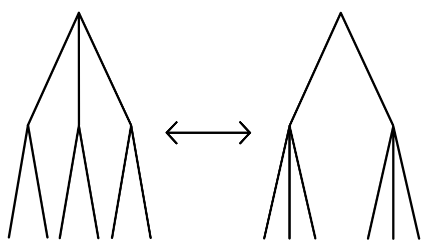

Figure 18: The two equivalent subtrees of Theorem 0.6 when $n = 1$ and $m = 2$.

Theorem 0.7. *The root caret of a tree can be written as an p–ary caret if and only if every the p-valence of every leaf in the tree is greater than zero.*

Presentations for $F(2,3)$

The infinite presentation has the following generators:

$$\{x_0, x_1, ..., y_0, y_1, ..., z_0, z_1, ...\}$$

where the x_is are simply the same generators as in $F(2)$. We can see the tree-pair diagrams for these generators in Figure 19. (The generators we refer to as z_i are the same as those which Stein refers to as y_i^+ in her paper).

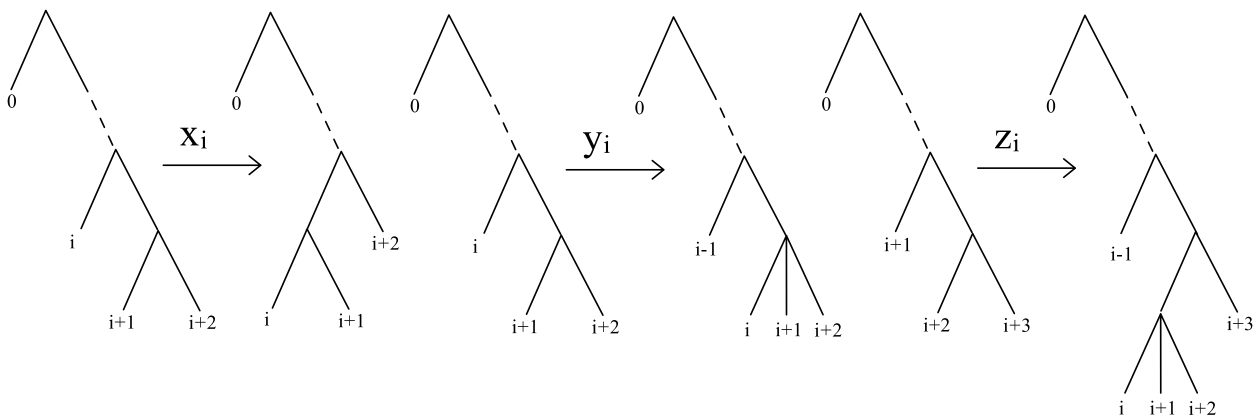

Figure 19: The generators for the standard infinite presentation of $F(2,3)$

The relators for this presentation are:

1. $x_j x_i = x_i x_{j+1}$
2. $y_j x_i = x_i y_{j+1}$
3. $z_j x_i = x_i z_{j+1}$
4. $x_j z_i = z_i x_{j+2}$
5. $y_j z_i = z_i y_{j+2}$
6. $z_j z_i = z_i z_{j+2}$

for $i < j$ and

1. $y_{i+1} z_i = y_i x_{i+1} x_i$
2. $x_i z_{i+1} z_i = z_i x_{i+2} x_{i+1} x_i$

for all i.

Normal Form

If we have a minimal tree-pair diagram representative of an element of $F(2,3)$, we can write down the normal form for that element.

Lemma 0.1. *The normal form of an element w in $F(2,3)$ can be written as:*

$$NF(w) = NF(w_+)NF(w_-^{-1})$$

where w_+ is the element of $F(2,3)$ with minimal tree pair diagram representative consisting of a negative tree of all right 2–ary carets and a positive tree identical to the positive tree in the minimal tree-pair diagram representative of w. We can define w_- similarly for the negative tree.

Definition (leaf exponent matrix) 0.2. *The leaf path for a leaf l_i in a tree is the string of carets on the directed edge from the root node to the leaf.*

Now we define a subpath of the leaf path which we will call the exponent path:
the string of carets consisting of all carets $\wedge_j$ on the leaf path of l_i such that

1. *$\wedge_j$ has its left edge on the directed edge from the root node to the leaf*
2. *and $\wedge_j$ is not a right caret or the root caret.*

The number of carets in this subtree is the leaf exponent of l_i.

The exponent path can be broken into subpaths which

1. *contain only one valence of carets and*
2. *contain all consecutive carets of that valence on that part of the exponent path.*

If we let $\lambda_{i,1}$ denote the subpath closest to the root and if we let $\lambda_{i,j}$ denote the adjacent subpath to $\lambda_{i,j-1}$ which is closer to the leaf l_i, then the leaf exponent matrix for l_i is

$$E_i = \begin{pmatrix} \epsilon_{i,1} & \cdots & \epsilon_{i,m} \\ d_{i,1} & \cdots & d_{i,m} \end{pmatrix}$$

where

$$\epsilon_{i,j} = \begin{cases} 2 \text{ if carets in the subpath } \lambda_{i,j} \text{ are 2–ary} \\ 3 \text{ if carets in the subpath } \lambda_{i,j} \text{ are 3–ary} \end{cases}$$

and where $d_{i,j}$ is the number of carets on the subpath $\lambda_{i,j}$. Clearly the leaf exponent of l_i will be equal to

$$\sum_{j=1}^{m} d_{i,j}$$

The leaf exponent matrix can be most easily understood by doing a few examples:

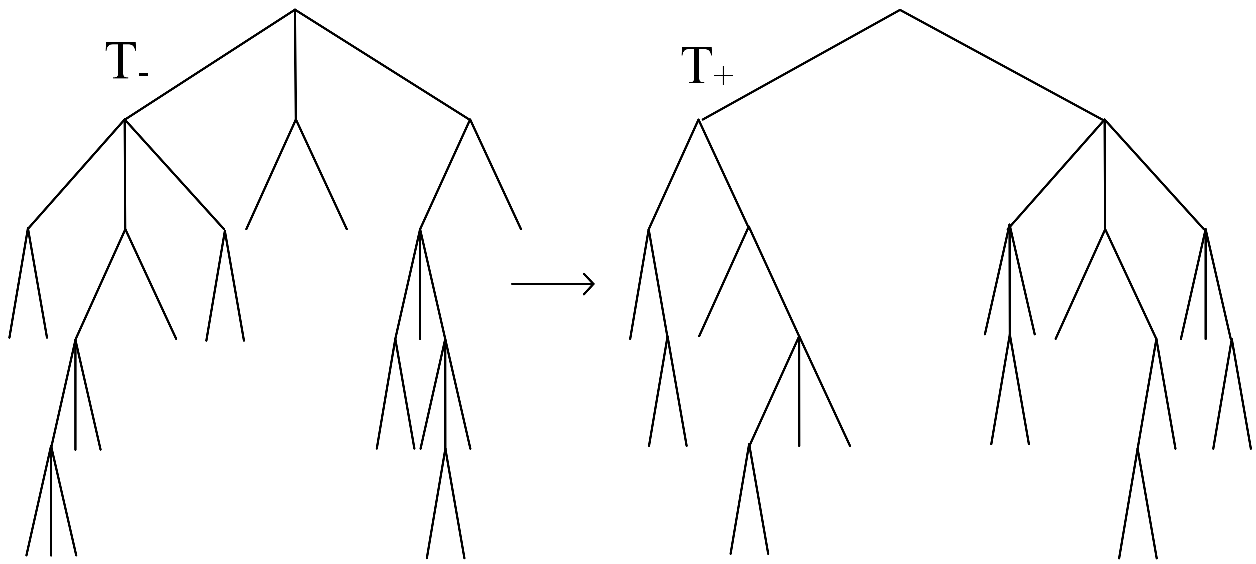

Figure 20: Tree-pair diagram representative of an element w of $F(2,3)$

We compute the exponent matrix for each leaf in the tree-pair diagram given in Figure 21:

For the positive tree we get:

$$E_0 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, E_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, E_3 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, E_4 = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}, E_8 = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

$$E_{10} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}, E_{13} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, E_{14} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For the negative tree we get:

$$E_0 = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, E_8 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, E_{10} = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

$$E_{12} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}, E_{15} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, E_{16} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Theorem 0.3. *For a positive element w of $F(2,3)$ the normal form is:*

$$NF(w) = \mathcal{B}_0 \cdots \mathcal{B}_k$$

where

$$\mathcal{B}_0 = y_{\beta_1} \cdots y_{\beta_N}$$

such that $\beta_{i+1} > \beta_i + 1$ *for all* $i \in \{1, ..., N\}$ *and where*

$$\mathcal{B}_i = (\gamma^{i,1}_{\alpha i,1})^{e_{i,1}} \cdots (\gamma^{i,s_i}_{\alpha i,s_i})^{e_{i,s_i}}$$

such that

$$\alpha_{1,j} \neq 0 \textit{ for all } j \in \{1, ..., s_1\}$$

and

$$\alpha_{i,1} = 0 \textit{ and } e_{i,1} = 1 \textit{ for all } i \in \{2, ..., k\}$$

and

$$\alpha_{i,j} \neq 0 \textit{ for all } j \in \{2, ..., s_i\} \textit{ and } i \in \{2, ..., k\}$$

and where for all $i \in \{2, ..., k\}$

$$\alpha_{i,j} \leq \sum_{l=1}^{j-1} e_{i,l} \cdot \left(\omega\left(\gamma^{i,l}_{\alpha_{i,l}}\right) - 1\right)$$

for all $j \in \{1, ..., s_i\}$.

where $\omega(\gamma_i)$ *is defined such that*

$$\omega(\gamma_i) = \begin{cases} 2 \textit{ if } \gamma_i = x_i \\ 3 \textit{ if } \gamma_i = z_i \end{cases}$$

Now we are ready to find the normal form for the element represented by the tree-pair diagram given in Figure 21:

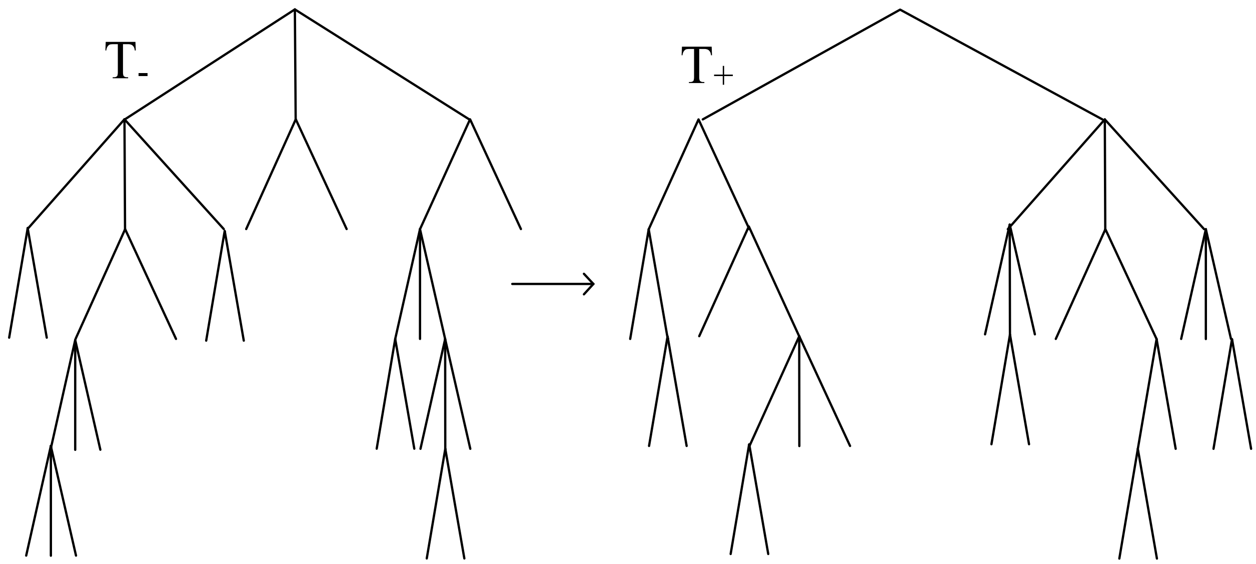

Figure 21: Tree-pair diagram representative of an element w of $F(2,3)$

$$\begin{aligned}
NF(w) &= NF(w_+)NF(w_-^{-1}) \\
&= y_1y_3z_1x_2x_5x_6^2x_0x_1z_2x_2x_0x_1(y_0x_1z_3x_3z_6x_7z_0x_1z_1^2x_7x_0)^{-1} \\
&= y_1y_3z_1x_2x_5x_6^2x_0x_1z_2x_2x_0x_1x_0^{-1}x_7^{-1}z_1^{-2}x_1^{-1}z_0^{-1}x_7^{-1}z_6^{-1}x_3^{-1}z_3^{-1}x_1^{-1}y_0^{-1}
\end{aligned}$$