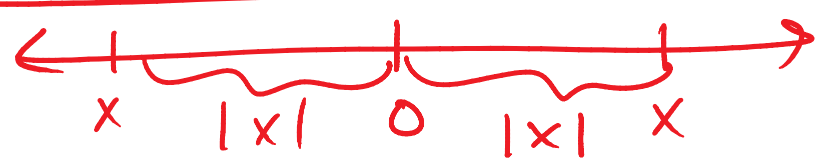
Absolute Value: Solving Equations

What is an absolute value?

Defn: The absolute value of x is its magnitude, or its distance from zero on the number line.

We write $|x|$ to denote the absolute value of x.



Note: Because absolute value is a distance, it is always positive.

If x is positive, then $|x| = x$.

If x is negative, then $|x| = -x$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Why are we interested in absolute value?

In real life, we may have distances or magnitudes that can only be positive.

Examples:

- How far apart are two cars?

$C_1 \cdot C_2$ $\quad |C_1 - C_2| =$ distance between cars

How do we solve absolute value equations?

If we want to solve for a variable (like x), but the variable is inside an absolute value sign, we need to find a way to rewrite the equation without the absolute value sign, so that we can "move" the variable around.

How can we get rid of an absolute value sign? OR, how can we replace an absolute value equation with equivalent equation(s) that don't have an absolute value sign?

If we have $\boxed{|A| = B}$ for some algebraic expressions A & B (where $B \geq 0$), Note! We must have $B \geq 0$, because an absolute value can never be negative!

Similarly, if we have $|A| = |B|$

Either A & B have the same sign, in which case $A = B$ or

A & B have opposite signs, in which case, $A = -B$ (or $-A = B$, which means the same thing)

So $|A| = |B|$ can be rewritten as

$A = B$ or $A = -B$,

Examples:

1) $|a| = 2.5$ Solve for a:

$\Rightarrow$ $\boxed{a = 2.5 \text{ or } a = -2.5}$

2) $-3 = |y|$ Solve for y:

Cannot rewrite this without the absolute value $\Rightarrow$ because this equation has set the absolute value equal to something negative

No solution! (Solution set is empty. $\{\}$)

3) $9 = |3-2b|$ Solve for b:

$9 = 3-2b$ or $-9 = 3-2b$

$-3 \quad -3$ $\qquad$ $-3 \quad -3$

$\Rightarrow \dfrac{6}{-2} = \dfrac{-2b}{-2}$ $\qquad$ $\Rightarrow \dfrac{-12}{-2} = \dfrac{-2b}{-2}$

$\boxed{-3 = b}$ or $\boxed{6 = b}$

Check: $9 = |3-2(-3)|$ $\qquad$ $9 = |3-2(6)|$

Check: $9 = |3 - 2(-3)|$ $\quad$ $9 = |3 - 2(6)|$

$\Rightarrow 9 = |3 + 6|$ $\quad$ $\Rightarrow 9 = |3 - 12|$

$\Rightarrow 9 = |9|$ ✓ $\quad$ $\Rightarrow 9 = |-9|$ ✓

4) $|x-7| = |2x-1|$ Solve for x:

Note: The whole side must be made negative!

$x-7 = 2x-1$ or $x-7 = -(2x-1)$

$-x \qquad -x$

$\Rightarrow -7 = x-1$

$+1 \qquad +1$

$\Rightarrow \boxed{-6 = x}$ ✓

or

$\Rightarrow x-7 = -2x+1$

$+2x \qquad +2x$

$\Rightarrow 3x-7 = 1$

$+7 \qquad +7$

$\Rightarrow \frac{3x}{3} = \frac{8}{3}$

$\Rightarrow \boxed{x = \frac{8}{3}}$ ✓

Check:

$|(-6)-7| = |2(-6)-1|$

$|-13| = |-12-1|$

$|-13| = |-13|$

$13 = 13$ ✓

$\left|\left(\frac{8}{3}\right) - \frac{7 \cdot 3}{1 \cdot 3}\right| = \left|\frac{2}{1}\left(\frac{8}{3}\right) - \frac{1 \cdot 3}{1 \cdot 3}\right|$

$\Rightarrow \left|\frac{8}{3} - \frac{21}{3}\right| = \left|\frac{16}{3} - \frac{3}{3}\right|$

$\Rightarrow \left|-\frac{13}{3}\right| = \left|\frac{13}{3}\right|$

$\Rightarrow \frac{13}{3} = \frac{13}{3}$ ✓

5) $|3x-7| = |1-3x|$ Solve for x:

$3x - 7 = 1 - 3x$
$-3x \qquad -3x$
$\Rightarrow -7 = 1 - 6x$
$-1 \quad -1$
$\Rightarrow \dfrac{-8}{-6} = \dfrac{-6x}{-6}$
$\Rightarrow \boxed{\dfrac{4}{3} = x}$

one solution only

or

$3x - 7 = -(1 - 3x)$
$3x - 7 = -1 + 3x$
$-3x \qquad -3x$
$-7 = -1 \Rightarrow$ not true
$\Downarrow$
no solution!

Check:

$\left|3\left(\frac{4}{3}\right) - 7\right| = \left|1 - 3\left(\frac{4}{3}\right)\right|$

$\Rightarrow \left|\frac{\cancel{3}}{1} \cdot \frac{4}{\cancel{3}} - 7\right| = \left|1 - \frac{\cancel{3}}{1} \cdot \frac{4}{\cancel{3}}\right|$

$\Rightarrow |4 - 7| = |1 - 4|$

$\Rightarrow |-3| = |-3|$

$\Rightarrow 3 = 3$ ✓

6) $|y-8| = |8-y|$ Solve for y:

$y-8 = 8-y$ or $y-8 = -(8-y)$

$-y \qquad -y$

$\Rightarrow -8 = 8-2y$

$-8 \quad -8$

$\Rightarrow \frac{-16}{-2} = \frac{-2y}{-2}$

$\Rightarrow \boxed{8 = y}$ or

$y-8 = -8+y$

$-y \qquad -y$

$-8 = -8$

always true $\Rightarrow$ solution set is

$\boxed{\text{all real numbers}}$

Since 8 is a real number, it is simpler to say:

$\boxed{\text{all real numbers}}$