Factoring Polynomials

What is factoring?

Taking a polynomial and rewriting as a product of several factors (polynomials).

We usually factor polynomials completely - we break them down into the simplest possible polynomials (none of the factors can be further broken down by factoring).

Ex: This is like factoring numbers into a product of prime numbers:

1) $18 = 3 \cdot 6$
$= 3 \cdot 2 \cdot 3$
$= \boxed{2 \cdot 3 \cdot 3}$ or $\boxed{2 \cdot 3^2}$

2) $10x^2 - 22x + 4 = (x-2)(10x-2)$

Why are we interested in factoring polynomials?

1) It can be used to simplify expressions or equations.

Ex: 1) $\frac{80}{124} = \frac{8\cdot 10}{4\cdot 31} = \frac{\overbrace{2\cdot 2}^{\div 4 = 1}\cdot 2\cdot 2\cdot 5}{\underbrace{2\cdot 2}_{\div 4 = 1}\cdot 31} = \frac{2\cdot 2\cdot 5}{31} = \boxed{\frac{20}{31}}$

2) $\frac{10x^2-22x+4}{8x^4-16x^3-3x^2+6x} = \frac{\cancel{(x-2)}(10x-2)}{\cancel{(x-2)}(8x^2-3)x}$

$= \boxed{\frac{10x-2}{(8x^2-3)x}}$

2) It can be used to solve polynomial equations:

Ex: $x^2-6x+8=0$ solve for x.

$\cancel{x^2-6x=-8}$ → How do we get x by itself here?

$(x-2)(x-4)=0$

one of these pieces must be 0, in order to get zero when we multiply.

$\boxed{x=2 \text{ or } x=4}$

What kinds of polynomials will we learn how to factor?

We will factor polynomials over the integers only. $\Rightarrow$ All the coefficients in our terms will be integers. (no fractions, no irrational numbers, no imaginary or complex numbers)

Ex: $(x^2-2) = (x+\sqrt{2})(x-\sqrt{2})$

(x^2-2): ✓ We leave it alone.

$(x+\sqrt{2})(x-\sqrt{2})$: not integers, irrational numbers

$(x^2+1) = (x+i)(x-i)$

(x^2+1): ✓ We leave it alone.

$(x+i)(x-i)$: not integers, imaginary numbers

How do we factor polynomials?

One option: trial + error

Another option

Look for patterns when we multiply polynomials.

Then we can use these patterns to identify polynomials that can be factored in a particular way.

Patterns when multiplying polynomials

Definition:

monomial = polynomial with 1 term

Ex: 3 $\quad x^2y$ $\quad -21a^4$ $\quad 47a^2b^3$

binomial = polynomial with 2 terms

Ex: $x-y$ $\quad x^2+6$ $\quad 3ab-b^2$

trinomial = polynomial with 3 terms

x^2+2x-9 $\quad a^2-2ab+3b^2$

Most of the things we will focus on factoring will be a product of:

- a monomial and another polynomial
- two binomials

Multiplying a Monomial by Another Polynomial – Patterns!

Example: 1) $3x(x^2 - 2x + 1) = 3x(x^2 + -2x + 1)$

$= 3x \cdot x^2 + 3x \cdot -2x + 3x \cdot 1$

$= \boxed{3x^3 + -6x^2 + 3x}$

$3x(x^2 + -2x + 1)$

2) $(x^2 - y^2)(3xy^3) = (x^2 + -y^2)(3xy^3)$

$= x^2 \cdot 3xy^3 + -1y^2 \cdot 3xy^3$

$= \boxed{3x^3y^3 + -3xy^5}$

$3xy^3(x^2 + -y^2)$

Pattern: (Factoring out a Monomial)

Find the biggest thing that divides evenly into every single term. We call that the greatest common factor.

$AB_1 + AB_2 + AB_3 + \ldots + AB_n$ → Divide each piece by the GCF which is A.

$= \underline{A}(B_1 + B_2 + B_3 + \ldots + B_n)$

↓
GCF

<u>Note</u>: Since <u>looking for a greatest common factor</u> is one of the easiest things to do, whenever we have a polynomial to factor, it makes sense to try this <u>first</u>.

Patterns when multiplying two binomials

Examples:

1) $(x^2+5)(2x-1) = (x^2+5)(2x+{}^-1)$

$= x^2\cdot(2x+{}^-1) + 5\cdot(2x+{}^-1)$

$= (x^2\cdot 2x + x^2\cdot{}^-1) + (5\cdot 2x + 5\cdot{}^-1)$

$= \boxed{(2x^3+{}^-x^2) + (10x+{}^-5)}$

GCF is x^2 GCF is 5

2) $(3x-1)(2x+5) = (3x+{}^-1)(2x+5)$

$= 3x(2x+5) + {}^-1(2x+5)$

$= 3x\cdot 2x + 3x\cdot 5 + {}^-1\cdot 2x + {}^-1\cdot 5$

$= 6x^2 + \underbrace{15x + {}^-2x} + {}^-5$

$= \boxed{6x^2+13x+{}^-5}$

3) $(a+b)(a-b) = (a+b)(a+{}^-b)$

$= a\cdot(a+{}^-b) + b\cdot(a+{}^-b)$

$= a\cdot a + a\cdot{}^-b + b\cdot a + b\cdot{}^-b$

$= a^2 + \underbrace{{}^-ab + ab}_{0} + {}^-b^2$

$= a^2 + 0 + {}^-b^2$

$= \boxed{a^2+{}^-b^2}$ or $\boxed{a^2-b^2}$

Pattern from Example 1:

To go backwards and factor, we need to group together the first two terms and the last two terms.

We then factor out the GCF from each group.

We can then treat these twos GCFs as coefficients & combine them into a single binomial coefficient.

We call this pattern **factoring by grouping**.

Examples:

1) $2a^3 - a^2 + 2a - 1 = \underbrace{(2a^3 + -a^2)}_{GCF = a^2} + \underbrace{(2a + -1)}_{GCF = 1}$

$= a^2\left(\frac{2a^3}{a^2} + \frac{-a^2}{a^2}\right) + 1(2a + -1)$

$= a^2\underbrace{(2a + -1) + 1(2a + -1)}_{\text{same variable part}}$

$= \boxed{(a^2 + 1)(2a + -1)}$ ✓

2) $x^3 - y^2 - x^2y + xy = \underbrace{(x^3 + -y^2)}_{GCF = 1} + \underbrace{(-x^2y + xy)}_{GCF = xy}$

$= 1\underbrace{(x^3 + -y^2) + xy(-x + 1)}_{\text{NOT the same variable part!}}$

STOP!

This grouping didn't work, so let's reorder + try a different grouping!

$= \underbrace{(x^3 + -x^2y)}_{GCF = x^2} + \underbrace{(xy + -y^2)}_{y}$

$= x^2\underbrace{(x + -y) + y(x + -y)}_{\text{variable part}}$ is the same!

$= \boxed{(x^2 + y)(x + -y)}$

Note: When we have **4 terms** factoring by grouping is a logical thing to try.

Pattern in Example 2:

Ex 2: $(3x-1)(2x+5) = (3x+-1)(2x+5)$

$= 3x(2x+5)+-1(2x+5)$ factoring by grouping

$= 3x\cdot 2x+3x\cdot 5+-1\cdot 2x+-1\cdot 5$

$= 6x^2+\underbrace{15x+-2x}_{\text{like terms}}+-5$

$= 6x^2+\underline{13x}+-5$ → we end up with 3 instead of 4 terms

This came from the <u>sum</u> of <u>two</u> like terms.

We have a polynomial of the form ax^2+bx+c; we need to find a way to <u>rewrite the bx part</u> as <u>the sum of two terms</u>.

We need to pick those terms so that we can then <u>factor by grouping</u>.

So how do we find these two pieces?

How do we break bx into two pieces?

Let's think about the general form we have when we multiply two binomials like the ones in example 2:
(Note: In math, $a \neq A$)

General Form:

$$(Ax+B)(Cx+D) = Ax(Cx+D) + B(Cx+D)$$
$$= Ax \cdot Cx + Ax \cdot D + B \cdot Cx + B \cdot D$$
$$= ACx^2 + \underbrace{ADx + BCx} + BD$$
$$= \underbrace{AC}_{a}x^2 + \underbrace{(AD+BC)}_{b}x + \underbrace{BD}_{c}$$

ax^2+bx+c
general form

Notice: $ac = AC \cdot BD = AD \cdot BC$
$b = AD + BC$

The two pieces we are looking for need to:
1) add up to b
2) multiply to get ac

Pattern: If we are try to factor a polynomial of the form

$ax^2 + bx + C$ (a, b, c are integers)

all we need to do is to find two numbers that:

1) add up to b

2) multiply to get ac

Then we rewrite bx as the sum of these two coefficients & factor by grouping

Examples: (break bx into two pieces and then factor by grouping)

ax^2+bx+c

$a=1$

$b=-1$

$c=-72$

1) $x^2-x-72=1x^2+-1x+-72$

We need to find two numbers that:

1) add up to -1

2) multiply to get $1\cdot -72 = -72$

What pairs of numbers can we multiply to get -72?

~~2, -36~~ or ~~-2, 36~~

~~3, -24~~ or ~~-3, 24~~

~~4, -18~~ or ~~-4, 18~~

~~6, -12~~ or ~~-6, 12~~

8, -9 or -8, 9

$8+-9=-1$ and $8\cdot -9=-72$

$$1x^2+-1x+-72$$

$$=1x^2+(8+-9)x+-72$$

$$=\underbrace{(x^2+8x)}_{GCF=x}+\underbrace{(-9x+-72)}_{GCF=-9}$$

$$=x(x+8)+-9(x+8)$$

equal

$$=\boxed{(x+-9)(x+8)}$$

ax^2+bx+c $\quad a=12$, $b=-19$, $c=-18$

2) $12x^2-19x-18=12x^2+-19x+-18$

So: I need to find two numbers that:

- add up to -19
- multiply to get $12\cdot -18=-216$

Factors of -216:

$-8, 27$ or $8, -27$

$8+-27=-19$

$8\cdot -27=-216$

$$12x^2+-19x+-18$$

$$=12x^2+\overbrace{(8+-27)}x+-18$$

$$=\underbrace{(12x^2+8x)}_{GCF=4x}+\underbrace{(-27x+-18)}_{GCF=-9}$$

$$=4x\underbrace{(3x+2)}+-9\underbrace{(3x+2)}$$

(equal!)

$$=\boxed{(4x+-9)(3x+2)}$$

Note! When we have something of the form ax^2+bx+c it makes sense to try this method, the ac-method.

Patterns in Example 3:

Ex 3: $(a+b)(a-b) = (a+b)(a+-b)$

$= a(a+-b) + b(a+-b)$

$= a\cdot a + a\cdot -b + b\cdot a + b\cdot -b$

$= a^2 + \underbrace{-ab + ab}_{\text{zero}} + -b^2$

$= a^2 + 0 + -b^2$

$= \boxed{a^2 + -b^2}$ or $\boxed{a^2 - b^2}$

Pattern:

When we have a difference of squares (something of the form $A^2 - B^2$), we can rewrite as the product of a sum and a difference:

$$A^2 - B^2 = (A+B)(A-B)$$

Examples:

1) $9x^2 - 49y^2 = (\overset{A}{\overset{\shortparallel}{3x}})^2 - (\overset{B}{\overset{\shortparallel}{7y}})^2$

$= \boxed{(3x+7y)(3x-7y)}$

2) $x^4 - 16 = (\overset{A}{\overset{\shortparallel}{x^2}})^2 - (\overset{B}{\overset{\shortparallel}{4}})^2$

$= (x^2+4)\underbrace{(x^2-4)}_{\text{difference of squares}}$ ✓

can be rewritten as a square!

$= (x^2+4)(x^2-2^2)$

$= \boxed{(x^2+4)(x+2)(x-2)}$

Note: Whenever we want to factor a polynomial that has two terms, it makes sense to check to see if it is a difference of squares

Note: Sum of squares: $\cancel{A^2+B^2}$

$(A+B)(A+B) = A(A+B) + B(A+B)$

$= A\cdot A \cdot A\cdot B + B\cdot A \cdot B\cdot B$

$= A^2 + AB + AB + B^2$

$= A^2 + 2AB + B^2$

Can't factor like difference of squares!

A few more patterns:

Sum of cubes:

$$\underline{A^3+B^3} = (A+B)\underline{(A^2-AB+B^2)}$$

Difference of cubes:

$$\underline{A^3-B^3 = (A-B)(A^2+AB+B^2)}$$

You check that this is true by multiply out the right side of each of these equations - you will get what's on the left side.

Ex: $8x^3-27y^6 = (\overset{A}{2x})^3-(\overset{B}{3y^2})^3$

$= (2x-3y^2)\underline{((2x)^2+(2x)(3y^2)+(3y^2)^2)}$

$= \boxed{(2x-3y^2)(4x^2+6xy^2+9y^4)}$

Techniques for Factoring Polynomials:

1) Find the greatest common factor and pull it out of each term

This is a good thing to try first.

2) Factoring by grouping

If it doesn't work with the first grouping that we choose, we may need to rearrange the terms and try a different grouping.

This is a good thing to try if we have four terms.

3) Factoring using the ac-method when the polynomial has the form ax^2+bx+c

This is a good pattern to look for when the polynomial has three terms.

4) Factoring using the difference of squares, difference of cubes, sum of cubes

$$A^2-B^2 \qquad A^3-B^3 \qquad A^3+B^3$$

Note: $\cancel{A^2+B^2}$ is not one of these forms

These are good patterns to look for when the polynomial has two terms.

Which techniques should we use?

1) $12x^3 - 27x$ $= 12x^3 + -27x$

$\underbrace{12x^3 - 27x}$ GCF $= 3x$

$= 3x(4x^2 + -9)$

$= 3x((2x)^2 - (3)^2)$ difference of squares

$= \boxed{3x(2x+3)(2x-3)}$

2) $x^3 - 4x^2 - 9x + 36 = (x^3 + -4x^2) + (-9x + 36)$ factoring by grouping

GCF $= x^2$ GCF $= 9$ -9

$= x^2(x + -4) + 9(-x + 4)$

almost the same

$-1(x + -4) = -x + 4$

$= x^2(x + -4) + -9(x + -4)$

equal

$= (x^2 + -9)(x + -4)$ difference of squares

$= (x^2 - 3^2)(x - 4)$

$= \boxed{(x+3)(x-3)(x-4)}$

3) $2x^2+4x-30 = 2x^2+4x+-30$

ax^2+bx+c

$a=2$
$b=4$
$c=-30$

Find two numbers that:

- add to get 4
- multiply to get $2\cdot -30 = -60$

$-6, 10$ or ~~$6, -10$~~

$-6+10=4$

$-6\cdot 10 = -60$

$$2x^2+4x+-30 = 2x^2+(-6+10)x+-30$$

$$= \underbrace{(2x^2+-6x)}_{GCF=2x} + \underbrace{(10x+-30)}_{GCF=10}$$

$$= 2x\underbrace{(x+-3)} + 10\underbrace{(x+-3)}$$

$$= \boxed{(2x+10)(x+-3)}$$

4) $\underbrace{3x^4+18x^2+27}_{\text{GCF is 3}} = 3(\underbrace{x^4+6x^2+9})$

Can I rewrite $\underline{x^4+6x^2+9}$

in the form ay^2+by+c ?

$x^2 = y$

$3(x^4+6x^2+9)$

$= 3(1(x^2)^2+6(x^2)+9)$

$3(y^2+6y+9)$

substitute $y=x^2$ at the end

$a=1$

$b=6$

$c=9$

Need 2 numbers that:

- add to get 6
- multiply to get $1\cdot 9 = 9$

3 and 3

$3+3=6$

$3\cdot 3=9$

$= 3((x^2)^2+(3+3)(x^2)+9)$

$= 3(\underline{(x^2)^2+3(x^2)+3(x^2)+9})$

$= 3((\underbrace{x^4+3x^2}_{GCF=x^2})+(\underbrace{3x^2+9}_{GCF=3}))$

$= 3(x^2(x^2+3)+3(x^2+3))$

$= \boxed{3(x^2+3)(x^2+3)}$