Wladis, BMCC/CUNY

Project 3: Curve Sketching

<u>For this project, the example, which you should use as a model for your own answers, have been handwritten – see attached page!</u>

For each of the following problems, graph $f(x)$, including: the correct domain and range; any intercepts; maximums or minimums; holes or cusps; horizontal, vertical or slant asymptotes; and correct curvature. Rotate through the rolls of *Prover*, *Explainer*, and *Checker*, so that each student plays a different role for each problem. The *Prover* should draw the graph step-by-step including doing all calculations needed to construct the graph, the *Explainer* should explain each step, and the *Checker* should check each step (using a different method than the *Prover* and *Explainer* if possible).

1. $f(x) = \frac{x^3}{x^2-9}$

2. $f(x) = 2x - 3x^{\frac{2}{3}}$

3. $f(x) = \frac{x^2-1}{x^2+25}$

Prover:	√/*	Explainer:	√/*	Checker:
Example E: Graph $f(x) = x + \frac{1}{(2x+1)^2}$				
① Domain: $2x+1 \neq 0$ $\Rightarrow \boxed{x \neq -\frac{1}{2}}$		If $x = -\frac{1}{2}$, $f(x)$ will be undefined, so at $x = -\frac{1}{2}$, $f(x)$ has a hole/jump or an asymptote. Since this singularity is not removable in this rational function, it must be a vertical asymptote.		
The singularity at $x = -\frac{1}{2}$ is not removable so: $\boxed{\text{vertical asymptote: at } x = -\frac{1}{2}}$				
② Intercepts:				
y-intercept: $y = (0) + \frac{1}{(2(0)+1)^2} = 1$ or $\boxed{(0,1)}$		$f(x)$ crosses the y-axis when $x=0$, so plugging $x=0$ into $f(x)$, we get $y=1$, which can be written as the point $(0,1)$.		
x-intercept(s): $0 = x + \frac{1}{(2x+1)^2}$ $\Rightarrow x = \frac{-1}{(2x+1)^2} \Rightarrow x(2x+1)^2 = -1$ $\Rightarrow 4x^3 + 4x^2 + x = -1$ $\Rightarrow 4x^3+4x^2+x+1=0$ $\Rightarrow 4x^2(x+1)+1(x+1)=0$ $\Rightarrow (4x^2+1)(x+1)=0$ $\Rightarrow 4x^2+1=0$ or $x+1=0$ ↳ no real # solution ↳ $x=-1$ or $\boxed{(-1,0)}$		$f(x)$ crosses the x-axis when $f(x)$ or $y=0$, so plugging 0 in for $f(x)$ and solving for x we get $x=-1$, which can be written as $(-1,0)$.		
③ Critical points: $f(x) = x + (2x+1)^{-2}$ so $f'(x) = 1 + 2\cdot -2(2x+1)^{-3}$ $= 1 - \frac{4}{(2x+1)^3}$ $f''(x) = -4\cdot 2\cdot -3(2x+1)^{-4}$ $= \frac{24}{(2x+1)^4}$		Critical points are the only possible points on the graph where $f(x)$ may change direction (e.g. from increasing to decreasing) or curvature (e.g. from concave up to concave down). This can happen only when $f'(x)$ is zero or undefined, or when $f''(x)$ is zero or undefined. So first we must find $f'(x)$ and $f''(x)$ in order to find the critical points.		

Due date:__________________

Prover:__________________ | **√/*** | **Explainer:**__________________ | **√/*** | **Checker:**__________________

$f'(x) = 0$ when $0 = 1 - \frac{4}{(2x+1)^3}$

$\Rightarrow 1 = \frac{4}{(2x+1)^3} \Rightarrow (2x+1)^3 = 4$

$\Rightarrow \sqrt[3]{(2x+1)^3} = \sqrt[3]{4}$

$\Rightarrow 2x+1 = \sqrt[3]{4} \approx 1.59$

$\Rightarrow \boxed{x \approx 0.295 \approx 0.30}$

$f'(x)$ is undefined when

$(2x+1)^3 = 0 \Leftrightarrow \boxed{x = -\frac{1}{2}}$

$f''(x) = 0$ when $0 = \frac{24}{(2x+1)^4}$

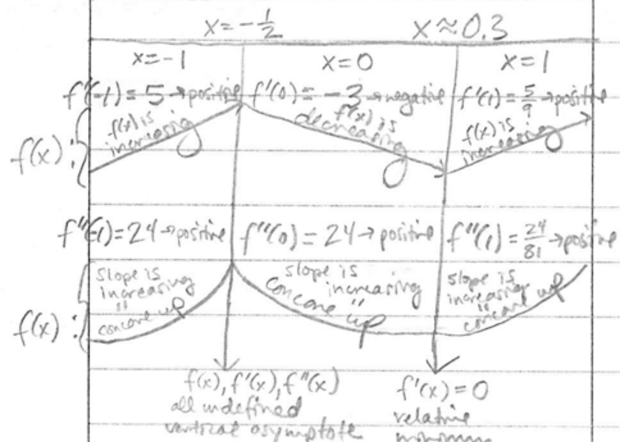
$\Downarrow$

no solution

$f''(x)$ is undefined when

$(2x+1)^4 = 0 \Leftrightarrow \boxed{x = -\frac{1}{2}}$

So critical x-values are: $X = -\frac{1}{2}, 0.3$



(4) Slant asymptote: at $y = x$

$\Downarrow$

no horizontal asymptote

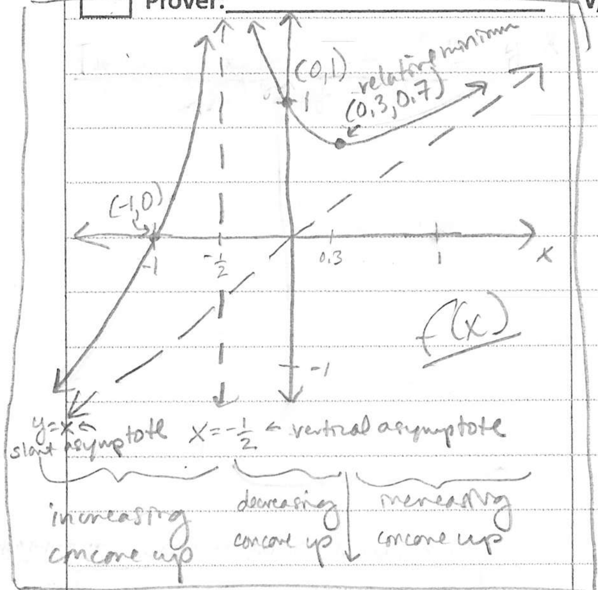
The only points where $f'(x)$ or $f''(x)$ are zero or undefined are where $x = -\frac{1}{2}$ and $x \approx 0.3$, so these are the only places where $f(x)$ might change direction or curvature. Since these are the only possible places where $f(x)$ could change curvature or direction, we only need to check one point in between each of these critical x-values. To the left of $x = -\frac{1}{2}$, we could choose -1. At $x = -1$, $f'(x)$ is positive, which means that the slope of $f(x)$ is positive to the left of $x = -\frac{1}{2}$. Similarly, we choose $x = 0$ between $x = -\frac{1}{2}$ and $x \approx 0.3$, and we choose $x = 1$ to the right of $x \approx 0.3$. At $x = 0$, $f'(x)$ is negative, so $f(x)$ is decreasing from left to right on that interval, and at $x = 1$, $f'(x)$ is positive, so $f(x)$ is increasing on that interval. Checking $f''(x)$, $f''(-1)$, $f''(0)$, and $f''(1)$ are all positive, so $f(x)$ is concave up on all three intervals. This is because when a positive slope is increasing, it is getting steeper, and when a negative slope is increasing, it is getting less steep (or less negative).

At $x \approx 0.3$, we have a relative minimum, since $f(x)$ has the shape ∪.

We have $f(x) = x + \frac{1}{(2x+1)^2}$, so as $x \to \pm\infty$, $\frac{1}{(2x+1)^2} \to 0$, which means that as $x \to \pm\infty$, $f(x) \to x$, or in other words, if we zoom out enough, $f(x)$ looks a lot like the graph of $y = x$.

Because there is a slant asymptote, there cannot be a vertical asymptote.

Prover:____________ | √/* | Explainer:____________ | √/* | Checker:____________



Final graph of $f(x)$.

We graph the intercepts $(0,1)$ and $(-1,0)$ and we draw dotted lines for the asymptotes $y=x$ and $x=-\frac{1}{2}$. Since to the left of $x=-\frac{1}{2}$, the graph is increasing & concave up, and since the graph must tend toward the lines $y=x$ and $x=-\frac{1}{2}$ without crossing them, the shape must be something like we have drawn here.

To the right of $x=-\frac{1}{2}$, we know that we have a relative minimum at $x \approx 0.3$. We find the exact point by plugging in $x \approx 0.3$ into $f(x)$ & we get $(0.3, 0.7)$ as the relative minimum. The graph must be concave up here, and it must be decreasing to the left of the minimum & increasing to the right of the minimum, and it must tend towards the lines $x=-\frac{1}{2}$ and $y=x$, in addition to passing through the points $(0,1)$ and $(0.3, 0.7)$.